ON THE INTERNAL REALIZATION OF POLYNOMIAL RESPONSE MAPS

By

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This work introduces a new class of nonlinear response maps, called polynomial maps. The natural internal (state-space) realizations of polynomial response maps are studied and a result is obtained on existence and uniqueness of canonical realizations. Various external finiteness conditions are related to corresponding ones of internal realizations. In particular, an investigation is made into the existence of input/output equations. The tools used are mainly algebraic-geometric. The subclass of bounded polynomial response maps is also introduced; in this case many realization results can be derived from automata and language theory.

CHAPTER I. OUTLINE OF MAIN CONTRIBUTIONS

In the present work we study the problem of realization of polynomial input/output maps. In this introduction we restrict ourselves to shift-invariant, scalar input/output maps defined over infinite fields, in order to present the definitions and results in a simple way. The development in the main text proceeds in greater generality.

We choose an infinite field k, which will be fixed throughout this section.

Let $\underline{\underline{S}}$ denote the set of all sequences of elements of k indexed by the integers and with support bounded on the left. In other words, $u(\cdot)$ in $\underline{\underline{S}}$ is a function $u(\cdot)\colon \underline{\underline{Z}} \to k$ for which there exists an integer t_o such that $u(t) = 0$ for all $t < t_o$. A (scalar) input/output map is then a map $f\colon \underline{\underline{S}} \to \underline{\underline{S}}\colon u(\cdot) \mapsto y(\cdot)$; f is (strictly) causal when, for all t in $\underline{\underline{Z}}$, the output $y(t)$ depends only on values $u(j)$ of inputs for $j < t$; f is shift-invariant if $u(\cdot) \mapsto y(\cdot)$ implies $(\sigma u)(\cdot) \mapsto (\sigma y)(\cdot)$, where σ is the shift operator defined by $(\sigma u)(t) := u(t - 1)$.

These concepts are standard. One of the contributions of the present work is the introduction of a new notion, that of a "polynomial" input/output map. Informally, this means that $y(t)$ is a polynomial function of the past input values $u(j)$, $j < t$.

In order to rigorously define polynomial input/output maps we need the concept of a Volterra series ψ: this is a formal power series in denumerably many variables $\xi_1, \xi_2, \xi_3, \ldots$ such that ψ is of finite degree in each variable separately. A causal, shift-invariant map is uniquely determined by specifying the dependence of $y(0)$ upon values of the input $u(t)$ for $t < 0$. So we can now say, more precisely, that a causal, shift-invariant input/output map f is polynomial iff there exists a Volterra series ψ_f such that the output $y(0)$ due to an input sequence $u(\cdot)$ is obtained by substituting $u(-j)$ for ξ_j into ψ_f and evaluating the expression thus obtained. This evaluation is well-defined because there are only finitely many nonzero $u(t)$ (by definition of $\underline{\underline{S}}$), and because ψ_f is a polynomial in each finite subset of variables.

The present formalism is able to represent a wide variety of behaviors. For example, consider f_1, defined by $\psi_{f_1} = a_1\xi_1 + a_2\xi_2 + \ldots$. Then an input $u(\cdot)$ produces an output

$$\sum_{j=0}^{\infty} a_j u(-j).$$

Thus f_1 corresponds to a linear system with impulse response sequence $a_1, a_2, \ldots$.

Another example is $\psi_{f_2} = \Sigma\, \xi_{j_1}\xi_{j_2}\cdots\xi_{j_r}$, the sum running over all $j_1 < j_2 < \ldots < j_r$ and all $r \geq 0$. Then f_2 corresponds to adding all possible products of past inputs.

The main problem we are interested in is the following. What are natural internal (i.e., state-space) representations for polynomial input/output maps? We are of course interested in representations with a certain amount of algebraic, geometric, and/or topological structure; otherwise the above question could be trivially answered via the "Nerode realization" method of automata theory. Further, we want to use our results to infer possible internal properties of a given "black box"; so the choice of structure should be directly related to properties of polynomial input/output maps.

Polynomial systems constitute a class of systems whose defining maps are always polynomial. A polynomial system Σ (provisional definition) has

(a) $X = k^n$ as its state-space (n = integer);

(b) state-transitions given by simultaneous first order difference equations

$$x(t + 1) = P(x(t), u(t)),$$

where $x(t) = (x_1(t), \ldots, x_n(t))$ and $P = (P_1, \ldots, P_n)$ is a polynomial function $k^{n+m} \to k^n$;

(c) an output map

$$y(t) = h(x(t)),$$

where h is a polynomial in n variables; and

(d) an initial state $x^{\#}$ which is an equilibrium state for the zero input:

$$P(x^{\#}, 0) = x^{\#}.$$

(The constraint on $x^{\#}$ to be an equilibrium state is dictated by our restriction to shift-invariant input/output maps; the specific choice of 0 as "equilibrium input" is just a matter of choice of coordinates in the input space.) Let us denote by P also the recursive extension of P to sequences of inputs, i.e.

$$P(x, v_1, \ldots, v_{n+1}) := P(P(x, v_1, \ldots, v_n), v_{n+1}).$$

Then Σ defines an input/output map $f_{\Sigma}: u(\cdot) \mapsto y(\cdot)$ by the rule

$$y(t) := h(P(x^{\#}, u(t_o), u(t_o + 1), \ldots, u(t - 1)),$$

where $t_o < t$ is any integer for which $u(\tau) = 0$ if $\tau < t_o$. Then f_{Σ} is clearly a polynomial input/output map because it is defined as a composition of polynomials. We may in fact exhibit ψf_{Σ} directly by the rule that the coefficient of a monomial $\xi_i^{i_1} \ldots \xi_t^{i_t}$ should be equal to the coefficient of the same monomial in the polynomial $h(P(x^{\#}, \xi_t, \ldots, \xi_1))$.

Thus we have defined a large class of systems whose input/output maps are polynomial. Such systems are appealing from both mathematical and system-theoretic reasons, because they can be realized by finite interconnections of adders, multipliers, amplifiers, and delay lines. In order to get a reasonably complete and general theory, however, it is necessary to go beyond polynomial systems. A larger class of systems, called k-*systems*, arises when we study the problem of obtaining "canonical" realizations of input/output maps. The theory to be developed will show that k-systems provide the right amount of generality for studying realizations of polynomial maps. We now motivate their introduction.

One of our main objectives is to obtain realizations which are "natural" or "canonical" in the sense of not depending on any information not implied by the input/output behavior. The class of candidates to be considered should have some fixed structure (like polynomial systems) so that the canonical system is recoverable just from the knowledge of its external behavior. The approach which has been highly successful with automata and linear systems consists in trying to construct realizations which are as "minimal" or "irredundant" as possible; see for instance KALMAN, FALB, and ARBIB [1969, Chapters 7 and 10] and EILENBERG [1974, Chapters 3, 12, and 16]. We shall adopt such a viewpoint here, beginning with polynomial systems, and we shall see how we are forced to introduce more general systems.

Let us consider the two-dimensional system

$$\Sigma_0 = \begin{cases} x_1(t+1) = x_1(t) + u(t), \\ x_2(t+1) = x_1(t)x_2(t) + x_1(t) + x_2(t), \\ y(t) = x_2(t), \end{cases}$$

with initial state 0. It is easy to see that it is possible to reach from 0 any state $\begin{pmatrix} a \\ b \end{pmatrix}$ in k^2, using in fact inputs of length not greater than three. There are, however, redundant states which behave identically in the sense that they cannot be distinguished by input/output experiments. They are of the form $\begin{pmatrix} a \\ -1 \end{pmatrix}$. Any other states can be pairwise distinguished from the data $(x_2, x_1x_2 + x_1 + x_2)$ resulting of the observation of the output at two consecutive instants. In order to obtain a system with no unobservable states, we must identify the states $\begin{pmatrix} a \\ -1 \end{pmatrix}$ for all a in k and we must then define appropriate "polynomial" transitions, compatible with the original P, on the quotient set thus obtained. To have a well-defined notion of "polynomial map" we must first endow our quotient set with a suitable notion of "coordinate system", i.e. we need to define in it a geometric structure.

But this structure may not correspond to a polynomial system.

It turns out that the input/output map f_{Σ_o} of the above system Σ_o admits no observable polynomial realization. This remains true for the weaker question of existence of polynomial realizations for which we only require the property of distinguishable reachable states. In other words, it is in general impossible to embed the "Nerode realization" of an input/output map in a polynomial system, even if f is the input/output map of a polynomial system.

The natural algebraic-geometric way to proceed consists in introducing the notion of (k-points of) an affine k-scheme, or, as we shall say for short, a k-space. Such a space consists of a topological space X together with an algebra of polynomial functions on X (a distinguished family of continuous functions on X subject to appropriate axioms). In particular, the spaces k^n become k-spaces when endowed with the "Zariski topology", whose closed sets correspond to subsets of k^n defined by polynomial equations; the polynomial functions on the k-space k^n are the usual polynomial functions in n variables. Thus our previous choice of state-spaces furnishes an (easy) example of k-spaces. Given two k-spaces X_1, X_2 there is a well-defined concept of polynomial map $P: X_1 \to X_2$; these are precisely those maps which when composed with the polynomial functions on X_2 give polynomial functions on X_1.

A k-system Σ is then defined by letting the state set X_Σ be an arbitrary k-space and letting the transition map $X_\Sigma \times k \to X_\Sigma$ and the output function $X_\Sigma \to k$ be polynomial maps. The fundamental observation is that the input/output maps of k-systems are polynomial.

Conversely, each polynomial input/output map can be realized by some k-system. This fact follows rather trivially once that k-spaces have been recognized as the proper state spaces. The proof relies on turning the space of input sequences into a k-space Ω in such a way that the notion of (polynomial) input/output map becomes precisely that of a polynomial map between k-spaces.

Having established k-systems as the class of systems to be considered, we return to the problem that motivated the introduction of k-spaces in the first place, namely, the existence of "observable" realizations.

We shall prove that in the new class of systems it is always possible to "reduce" a given system to one all of whose states can be distinguished by input/output experiments. Nevertheless, this does not settle the question of observability. It was noticed already in SONTAG and ROUCHALEAU [1975] that there exist input/output maps having realizations Σ_1, Σ_2 both of which are reachable and observable but such that Σ_1 and Σ_2 are nonisomorphic (as k-spaces). In fact, the example given in the above reference has $X_1 = k$ while X_2 is a curve with a singularity, a very different kind of k-space. The difficulty lies in the concept of observability itself. This notion is usually defined by the intuitive requirement that different states be "distinguishable by processing the input/output data". The precise notion in this context is that different states should be distinguishable by an algebraic processing of the input/output data. This point of view leads to the definition of algebraic observability, which turns out to be the proper notion in our context.

The next step in our program for obtaining a "canonical" realization of a given input/output map is to construct an observable realization all of whose states are reachable from the initial state. Here we run into a new problem: the reachable set of an arbitrary system is not necessarily a k-space. For instance, let us consider a two-dimensional system with transitions defined by

$$x_1(t+1) = u(t),$$

$$x_2(t+1) = x_2(t)u(t) + x_2(t) + u(t),$$

and zero initial state. The reachable set fails to contain the points $\begin{pmatrix} -1 \\ b \end{pmatrix}$, $b \neq -1$.

This difficulty can be easily eliminated. Our ultimate goal is not to obtain reachable and observable realizations but rather to construct "natural" realizations. It is therefore enough to observe that (for continuity reasons), the dynamical properties of the reachable part of a system Σ uniquely determine the dynamical properties of the closure (in the topology of the k-space X_Σ) of the set of reachable states. (In the above example the closure corresponds to the whole plane.) We shall say that Σ is *quasi-reachable* if the closure of the reachable states is X_Σ. The closure of the subset of reachable states is always a k-space invariant under the action of inputs. So a quasi-reachable realization can always be obtained from an arbitrary realization. If we begin with a polynomial system, the closure of the reachable set is a very special type of k-space namely, an *algebraic variety*. It is natural therefore to generalize our preliminary definition of polynomial systems to include the case in which X_Σ is a variety (not necessarily k^n). In other words, *a polynomial system is given by a finite set of simultaneous polynomial difference equations together with a set of polynomial constraints on the state variables*.

We shall say that a k-system is *canonical* if it is quasi-reachable and algebraically observable. One of the main results of this work is then: *Every input/output map* f *admits a canonical realization* Σ_f *and any other canonical realization of* f *is isomorphic to* Σ_f. We have thus attained our goal of determining a natural class of state representations for polynomial input/output maps.

The result on existence and uniqueness of canonical realizations must be complemented by a discussion of finiteness conditions. In principle, there is of course no guarantee that the state-space X_f of Σ_f is in any sense "finite dimensional".

We have chosen the "transcendence degree" notion of dimension out of the many possible definitions of dimension of k-spaces. The *dimension* of a system Σ is then the dimension of X_Σ. Informally, the dimension of Σ counts the "degrees of freedom" in the state space. In the particular case of polynomial systems the dimension is what one would intuitively expect. For instance, if X_Σ is the "cusp" $\{(x_1, x_2) \in k^2 \mid x_1^3 = x_2^2\}$, then $\dim \Sigma = 1$.

We shall say that a given system Σ is almost polynomial when X_Σ can be obtained as a "quotient" of some space k^n (the terminology "quotient" is not quite precise here, since we shall have to admit in general the existence of some points besides those representing the equivalence classes of points of k^n). The name "almost polynomial" is due to the fact that in this case X_Σ can be expressed as a union of a variety and a lower-dimensional subset.

A central contribution of our work is: The input/output map f has a finite-dimensional realization if and only if Σ_f is an almost-polynomial system if and only if f satisfies an algebraic difference equation, i.e., if and only if there exists an integer s and a polynomial E in $2s + 1$ variables such that

$$E(y(t),\ y(t-1),\ \ldots,\ y(t-s),\ u(t-1),\ \ldots,\ u(t-s)) = 0,$$

for all input/output pairs $u(\cdot)$, $y(\cdot)$. Up to constant multiples there is a unique irreducible equation $E = 0$ of minimal order s satisfied by f. We shall also prove that, if f satisfies some algebraic difference equation, then f satisfies as well an equation $\hat{E} = 0$ linear in $y(t)$.

As a simple illustration of the above results, let $f := f_{\Sigma_o}$, where Σ_o is the system, introduced before,

$$\Sigma_o = \begin{cases} x_1(t+1) = x_1(t) + u(t), \\ x_2(t+1) = x_1(t)x_2(t) + x_1(t) + x_2(t), \\ y(t) = x_2(t), \end{cases}$$

with zero initial state. The canonical state-space X_f is, as a set, the union of the singleton $\{*\}$ and the subset $U := \{x_2 \neq -1\}$ of k^2. The transition and output maps of Σ_f are those induced by the projection $T\colon k^2 \to X_f\colon x \mapsto x$ if x is in U, $x \mapsto *$ if $x_2 = -1$. (These are polynomial maps for a suitable k-space structure on X_f.) The irreducible equation of minimal order satisfied by f is

$$[y(t-2)+1][y(t)+1]-[y(t-1)+1]^2-$$

$$-[y(t-1)+1][y(t-2)+1]u(t-2)=0.$$

In the "classical" case of linear systems it is well known that a system is canonical iff it is a minimal-dimensional realization of its input/output map. This result does not generalize directly to the present situation. A counterexample is given by the system (with $X = k^1$)

$$\Sigma = \begin{cases} x(t+1) = u(t), & x^\# = 0, \\ y(t) = x^2(t). \end{cases}$$

Clearly, Σ is not canonical, because all pairs of states $\{a, -a\}$ are indistinguishable. However Σ is minimal, since it has dimension 1.

The proper treatment of the above minimality question is through the concept of weakly canonical realizations. We shall say that Σ is weakly canonical when it is quasi-reachable and (in a sense to be made precise) "almost all" states are indistinguishable of only finitely many other states. The example in the previous paragraph is therefore weakly canonical, since in fact each state is indistinguishable of only one other state. Let k be either the field of real numbers or an algebraically closed field. We prove that a realization Σ of a polynomial input/output map f is of minimal dimension among all realizations of f if and only if Σ is weakly canonical. Over any field k, canonical realizations are minimal.

The question of deciding when Σ_f is in fact a polynomial system (i.e. X_f is a special kind of k-space: a variety) can be answered theoretically via the introduction of the observation algebra A_f of the input/output map. This is a k-algebra which is canonically associated to any given f. We prove that Σ_f is a polynomial system precisely when A_f is finitely generated as a k-algebra. Further, the smallest n for which Σ_f can be embedded in a system of n simultaneous polynomial difference equations is equal to the minimal possible cardinality of sets of generators of A_f. Unless A_f is isomorphic to a polynomial ring, n is not equal to the dimension of Σ_f. We shall also prove that Σ_f *is a polynomial system when* f *satisfies an input/output equation of the type*

$$a(u(t-1), \ldots, u(t-s))y(t)^r + \hat{E} = 0,$$

where $y(t)$ appears in $\hat{E}$ with degree less than r. Thus if, for instance, f is known to satisfy a regression equation, the realization theory of f can in principle be carried out without introducing the concept of k-spaces. Even in this special case, however, the general theory is needed in order to understand the meaning of the special hypothesis.

One of the main results is valid for input/output maps defined over fields k which contain the rational numbers. The result states that f *has a finite realization if and only if the Jacobian matrices in a certain sequence* $J_1(f), J_2(f), \ldots$ *have a uniformly bounded rank*. For a trivial example, we point out that when f is linear the matrix $J_n(f)$ is precisely the n-th principal minor of the behavior (Hankel) matrix of f.

All the results presented up to this point are proved later for *multivariable* polynomial input/output maps, for which both the inputs and outputs are vector-valued.

Proofs of the preceding results use tools of algebraic geometry. In other words, we use the "theory of polynomials" in the study of arbitrary polynomial input/output maps.

The second part of this work deals with a broad class of *bounded* (polynomial) input/output maps, whose study can be "linearized". This linearization permits us to obtain sharper statements. The isolation of the class of bounded maps is one of the contributions of this work.

Bounded maps f are defined as follows. Recall that ψ_f has a finite degree d_j in each variable ξ_j. We say that f is *bounded* when the degrees d_j are bounded independently of j. In other words, there exists an integer d such that no input is raised to a power higher than d. There are no restrictions on products between inputs at different instants and/or different channels. It is at first surprising that the concept of bounded map includes as particular cases all those families of maps for which a satisfactory realization theory has been developed in the past. For instance, linear systems, internally-bilinear systems (BROCKETT [1972], ISIDORI and RUBERTI [1973], ISIDORI [1973, 1974], FLIESS [1973, 1975], D'ALLESSANDRO, ISIDORI, and RUBERTI [1974], and others) give rise to bounded maps. (Internally-bilinear systems are those whose internal map is bilinear in the state and input and whose output map is linear. No products of inputs at same instants are performed by such systems, so $d := 1$ bounds all d_j.) Multilinear input/output maps (KALMAN [1968, 1976]) are also included. (Such maps allow products of inputs only in different channels, so that $d := 1$ is again a bound.)

We prove that if a bounded input/output map is at all realizable by a finite dimensional k-system, then it is also realizable by an (observable) state-affine system. The latter are (polynomial) systems with $X_\Sigma = k^n$ whose defining equations take the special form

$$x(t + 1) = F(u(t))x(t) + G(u(t)), \quad x^\# = 0,$$

$$y(t) = Hx(t).$$

where $F(\cdot)$ and $G(\cdot)$ are polynomial matrices and H is a linear map. The characteristic feature of state-affine systems is the linear occurrence of the state variable.

The above realizability result establishes state-affine systems as a very useful and natural class of systems with respect to bounded maps. Input/output maps realizable by state-affine systems (equivalently, finitely realizable bounded maps) are precisely those whose "observation space" (a linear space directly associated to the map) is finite-dimensional. These and other results indicate that state-affine systems play an "approximation" role in the discrete theory similar to the role of internally-bilinear systems in the continuous-time context (see for instance, FLIESS [1974, 1975] and SUSSMAN [1975]).

We then restrict our attention to realizations by state-affine systems. Canonical realizations can now be obtained (for bounded maps) without recourse to k-spaces. In fact, it is now natural to define *span-canonical* state-affine systems as observable systems such that the *linear span* of the reachable states is the full state-space k^n. We then prove that *span-canonical realizations of a given bounded finitely realizable* f *always exist*. Further, *any two such realizations can be related by a linear change of coordinates in the state-space*. Finally, *a realization is span canonical if and only if its dimension* n *is smallest possible among all state-affine realizations of the same input/output map*.

The above-mentioned results are proved by first associating to the bounded map f the *exponent* formal power series φ_f obtained directly from the Volterra series ψ_f. As opposed to ψ_f, the exponent series is a power series in *noncommutative* variables. The transformation $\psi_f \mapsto \varphi_f$ permits the explicit consideration of dynamics. We then remark that state-linear realizations are in a one-to-one correspondence with *representations* of φ_f. (The concept of representation of a noncommutative power series was introduced by SCHÜTZENBERGER [1961] as a generalization of automata-theoretic ideas, and has been rediscovered since by many authors, notably in the context of stochastic automata. Representations have been called *sequential systems* by CARLYLE and PAZ [1971] and *linear-space automata* by TURAKAINEN [1972]. A fairly complete account of representations, also called "automata with multiplicities", may be found in EILENBERG [1974]. The notion of representation which we use is in fact a minor variation of that in the literature.) The idea of associating representations to systems is not totally original, since an analogous method was used by FLIESS [1973] to study the special case of internally-bilinear systems. We give a brief but self-contained exposition of those results on representations which are relevant to our work.

An interesting observation is that, under the above one-to-one correspondence, span-reachability and observability for state-affine systems corresponds precisely to (automata-theoretic) reachability and observability for representations. Realizability of f can then be studied via the *behavior* (Hankel) *matrix* $\underline{\underline{B}}(f)$ of φ_f, and minimal state-affine realizations can be obtained by operating on $\underline{\underline{B}}(f)$, using the methods developed for representations by FLIESS [1972, 1975]. We give such a realization procedure, which generalizes and unifies known algorithms for linear and for the various kinds of bilinear systems. An interpretation of $\underline{\underline{B}}(f)$ follows from the remark that the observation space of f is isomorphic to the row space of $\underline{\underline{B}}(f)$.

We sharpen the result on algebraic difference equations by proving that *a bounded map is finitely realizable if and only if it satisfies an input/output difference equation which is linear in the output*. This is a new result even in the (very special) cases of internally-bilinear systems and multilinear input/output maps.

SCHÜTZENBERGER [1961] gave a generalization to power series of Kleene's theorem: *A language* L *is recognizable by a finite automaton* *if and only if* L *can be described by a regular expression*. This generalization can be applied to φ_f via the above correspondence between state-affine systems and representations. The conclusion is that f *has a state-affine realization if and only if* φ_f *is rational* i.e., if and only if φ_f can be obtained from polynomials by a finite number of additions, multiplications, and inversions. As a consequence, it becomes possible to apply the standard calculus of interconnections of automata (see, for instance, EILENBERG [1974]) to find φ_f (and therefore ψ_f) from any state-affine realization of f, and, viceversa, to construct realizations given rational expressions for φ_f.

We also define the subclass of *finite* maps f, corresponding to the restriction that the *total degree* of ψ_f should be finite. We show that the span canonical realization of such maps can be decomposed as a cascade of linear systems and memory-free nonlinearities. The existence of such decompositions characterizes finite maps.

Research in a new field is bound to suggest a wealth of open questions and new directions of investigation. In attacking the realization theory of nonlinear systems, the present work is no exception to that hope.

One of the byproducts of an algebraic study of systems is of course the development of algorithms for system analysis and design. In the case of bounded maps and state-affine systems, we use linear-algebraic techniques in constructing canonical realizations; these methods are a rather simple generalization of the classical Hankel matrix technique used so successfully in linear system theory. Finite dimensionality of the observation space is responsible for the linear-algebraic character of the study of bounded maps. This means that a nonlinear computational technique is indispensable as soon as nonlinear feedback is present in a system. An important question is, then: How effective are calculations with fields, algebras and polynomials? From its origins until (historically) not long ago, algebra remained to a great extent a computational discipline. The development of "modern" algebra [or the modern development of algebra] has shifted the emphasis towards generality and abstraction, permitting both the solution of heretofore unsolvable problems and the understanding of deep questions which can only now be even formulated in a rigorous way. Many questions of effective calculation have thus been left aside of the mainstream of algebra; a development which is particularly unfortunate in view of the advent of the digital computer. However, there are now signs of a trend toward the effectivization of various basic algebraic constructions. Some of these constructions can be used to solve system-theoretic questions. For instance, SEIDENBERG [1971] has worked on effective versions of Hilbert's Basis Theorem, and his results find an immediate application to questions of observability (SONTAG and ROUCHALEAU [1975]). The posthumous work of ROBINSON [1975] (see also CROSSLEY and NERODE [1975]) represents a promising approach to questions of computability in algebra, attacking such questions from the point of view of mathematical logic (model theory), but most of the detailed work remains to be done.

Of course, there is a large number of classical results, dealing with resultants and derivatives, and sometimes referred to by the label elimination theory, which permit the effective verification of certain conditions; our Jacobian criterion for finite-dimensional realizability involves a simple application of such results. It would certainly be of interest to explicitely compute the form of similar criteria for other problems; for instance, quasi-reachability of a polynomial system is also expressible by a Jacobian condition.

Many theoretical algebraic problems are suggested by the present work. For instance, as a rule system-theoretic questions depend for their clarification upon the development of a real (as opposed to complex) algebraic geometry. The papers of WHITNEY [1957] and of DUBOIS and EFROYMSON [1970] are among the few works in this area. In fact, the study of points in more arbitrary fields (e. g., the rational numbers) is needed from our viewpoint. For instance, the question of the validity, over $k \neq$ reals or k not algebraically closed, of the theorem: "A realization is minimal if and only if it is weakly canonical", leads to (unsolved) problems of an arithmetic, rather than geometric, nature. Another "pure" algebraic problem generated by our research is that of studying the structure of the "almost varieties" or "quotients of varieties" which appear as our canonical state-spaces. A thorough understanding of such spaces is an essential prerequisite to the practical application of the most general of our results, i. e., those dealing with the case in which Σ_f is not a polynomial system.

There are many problems which probably do not require finding new algebraic results. A typical open question of this type is: If k is algebraically closed, is the reachable part of every polynomial system actually reachable in bounded time?

Insofar as we have attacked the realization problem using methods not standard in system theory, there arises the possibility of applying the same methods to the study of other system-theoretic questions. Some steps toward this broader goal were taken in SONTAG and ROUCHALEAU [1975], but here again most of the work remains to be done.

CHAPTER II. ALGEBRAIC PRELIMINARIES

In this chapter we shall briefly discuss some basic notions of algebraic geometry which are used in the sequel. The main object to be introduced is the set of k-points of an affine k-scheme (k = field); we shall simply call this object a "k-space".

The study of k-spaces is per se not included in standard texts in algebraic geometry; usually one studies instead the set of all points of a scheme and then tries to deduce special properties of the k-points. For instance, the study of finitely generated reduced schemes over the reals $\underline{\underline{R}}$, i.e. the study of solutions of polynomial equations with real coefficients

$$(*) \qquad P_i(x_1, \dots, x_n) = 0, \quad i = 1, \dots, r,$$

focuses on the complex solutions $(x_1, \dots, x_n)$ in $\underline{\underline{C}}^n$ of (*) rather than on the real solutions. This approach has proved highly appealing, since statements concerning the set of complex solutions do not have to elucidate certain exceptional or degenerate cases. In fact, it is customary to proceed a step further and embed the corresponding problem in projective space.

To infer the nature of the set of k-points from the properties of the entire scheme is not always a straightforward matter; it may involve, in fact, nonalgebraic (e.g. differential-geometric) arguments.

For purposes of this exposition we have adopted the procedure of defining k-spaces directly. We shall give here the definitions and the main results needed later. With the exception of some trivial statements, no proofs will be given for those facts for which a precise reference is available (and given). There is unfortunately no single source for the results quoted. We rely mainly on BOURBAKI [1972] and DIEUDONNE [1974]. Except for some matters of style and emphasis, no original contributions appear in this chapter.

1. k-*Reduced Algebras*.

Let k denote an arbitrary but infinite field, to be fixed throughout the discussion. Recall that a (*commutative*) k-*algebra* (or, simply, an *algebra*) is a pair (A, φ), where A is a commutative ring with identity and $\varphi\colon k \to A$ is a ring homomorphism with $\varphi(1) = 1$. We shall denote such algebras by the corresponding ring A and identify k with its (isomorphic) image $\varphi(k)$. Thus the scalar product $r.a$, $r \in k$, $a \in A$, is the multiplication in A. The field k may always be viewed as a k-algebra, with φ = identity.

A *homomorphism of* k-*algebras* $\mu\colon A \to B$ will mean a homomorphism whose restriction to $\varphi(k) = k$ is the identity.

We adopt the following notation conventions:

(i) the first few upper-case Latin letters $A, B, C, \ldots$ denote k-algebras;

(ii) $A \otimes_k B$ or simply $A \otimes B$ is the tensor product algebra;

(iii) if A is an integral domain, then $Q(A)$ denotes the quotient field of A;

(iv) $k[T_1, \ldots, T_r]$ denotes the ring of polynomials in r variables over k; when $r = 1$ we write simply $k[T]$;

(v) "homomorphism" will *always* mean k-algebra homomorphism;

(vi) Hom (A, B) denotes the set of all homomorphisms $A \to B$.

(1.1) DEFINITION. *A* k-*ideal* M *of a* k-*algebra* A *is the kernel of a homomorphism* $A \to k$. *The* k-*radical* $\mathrm{rad}_k A$ *of* A *is the intersection of all* k-*ideals of* A.

Let M be a k-ideal of A. Since A is a k-algebra, $A/M \simeq k$ is a field, so M is maximal, but *not every maximal ideal of* A *is a* k-*ideal*. For instance, let $k = \mathbb{R}$ and $A = \mathbb{R}[T]$. Then the ideal M generated by $x^2 + 1$ is maximal (because $x^2 + 1$ is an irreducible polynomial) but $A/M \simeq \mathbb{C} \neq \mathbb{R}$. In the particular case in which A is finitely generated and k is algebraically closed, all maximal ideals are k-ideals; this is a consequence of Hilbert's Nullstellensatz; see BOURBAKI [1972, V.3.3, Proposition 2].

There is a bijective correspondence between k-ideals and homomorphisms $A \to k$. Indeed, let $\mu, \nu: A \to k$ and suppose that $\ker \mu = \ker \nu$. Take any x in A. Since $\mu(x)$ is in $k \subset A$ and μ is the identity on k, $x - \mu(x)$ belongs to $\ker \mu = \ker \nu$. Thus $0 = \nu(x - \mu(x)) = \nu(x) - \mu(x)$. So the maps μ and ν are equal.

Let X be a set. The set k^X of all functions $X \to k$ is a k-algebra under the pointwise operations, the constant functions constituting the subring isomorphic to k. A subalgebra of k^X is called an algebra of functions on X.

(1.2) **LEMMA.** The following statements are equivalent:

(a) $\mathrm{rad}_K A = \{0\}$.

(b) A is isomorphic to an algebra of functions.

PROOF. (b) implies (a). Let A be identified with a subalgebra of k^X. For each x in X, the evaluation map

$$e_x: k^X \to k: \varphi \mapsto \varphi(x),$$

restricted to A is a homomorphism, hence $\ker e_x|A$ is a k-ideal. Clearly then $\mathrm{rad}_K A \subseteq \bigcap \{\ker e_x, \ x \text{ in } X\} = \{0\}$.

(a) implies (b). Let $X(A)$ denote the set of all homomorphisms $\mu: A \to k$. Define $\iota: A \to k^{X(A)}$ by evaluation:

$$\iota(a)(\mu) := \mu(a).$$

Then ι is a homomorphism; moreover, it is in fact one-to-one. To prove this, assume that $\iota(a) = 0$, i.e. $\iota(a)(\mu) = \mu(a) = 0$ for some a in A and for all μ in $X(A)$. Then a is in the kernel of every $\mu: A \to k$, i.e. in the k-radical of A, which is 0 by hypothesis. So $a = 0$. Therefore $A \simeq \iota(A)$. □

Let us observe the duality implicit in the preceeding arguments. Homomorphisms $A \to k$ may be viewed as *points* on which elements of A act by evaluation.

This duality is fundamental in algebraic geometry.

(1.3) DEFINITION. *An algebra satisfying the conditions in* (1.2) *is called a* k-*reduced algebra*.

For every k-algebra A, the quotient ring $A/\mathrm{rad}_K A$ is k-reduced.

An important fact related to (1.3) is Hilberts's Nullstellensatz, which can be phrased as follows: *A finitely generated algebra* A *over an algebraically closed field* k *is* k-*reduced if and only if* A *has no nonzero nilpotents*.

A recent generalization of this celebrated result (DUBOIS [1967], DUBOIS and EFROYMSON [1970]) is the following: *A finitely generated algebra* A *over a maximally ordered field* k (*e.g.* $k = \mathbf{R}$) is k-*reduced if and only if for any* x_i *in* A *the relation* $\Sigma_{i=1}^n x_i^2 = 0$ *implies* $x_i = 0$ *for all* i.

The main idea in what follows is to view the elements of a k-reduced algebra A as functions on $X(A)$. Take x in $X(A)$, a in A. Viewed as a function $\iota(a)$ on $X(A)$, a has the value $\iota(a)(x) = x(a)$ at x. Except when discussing certain delicate points, we shall therefore identify A with its image under ι and so we shall write $\iota(a)$ as a and $\iota(a)(x)$ as $a(x)$. The fact that x is a homomorphism means that the algebra operations in A are now represented as pointwise operations: $(ab)(x) = a(x)b(x)$.

It is worth keeping in mind the following:

(1.4) EXAMPLE. Let $A := k[T_1, \ldots, T_n]$. Since k is infinite, a polynomial $a(T_1, \ldots, T_n)$ can be identified with the polynomial function $k^n \to k$: $(x_1, \ldots, x_n) \mapsto a(x_1, \ldots, x_n)$. Thus, by (1.2), A is k-reduced. A homomorphism μ: $A \to k$ is completely determined by giving values of

$\mu(T_1), \ldots, \mu(T_n)$ in k. Conversely, for any choice of $(x_1, \ldots, x_n)$ in k^n there is a homomorphism $\mu: A \to k$ defined by $\mu(T_i) = x_i$. Thus one identifies $X(A)$ with k^n and $\iota: A \to k^{X(A)}$ with the assignment: polynomial ↦ polynomial function. □

(1.5) LEMMA (*canonical factorizations*). *Let* $\tau: A \to B$, *where* B *is k-reduced*.

(i) *Then there exists a factorization*

$$\begin{tikzcd}
A \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & B \\
& C \arrow[ur, "\nu"'] &
\end{tikzcd}$$

where μ *is onto,* ν *is one-to-one, and* C *is k-reduced*.

(ii) *Further,* let two factorizations $A \xrightarrow{\mu_1} C_1 \xrightarrow{\nu_1} B$ *and* $A \xrightarrow{\mu_2} C_2 \xrightarrow{\nu_2} B$ *of* τ *be given, with* μ_1 *onto and* ν_2 *one-to-one*. *Then there exists a unique* $\eta: C_1 \to C_2$ *such that the following diagram commutes*:

$$\begin{tikzcd}
& C_1 \arrow[dd, "\eta"] \arrow[dr, "\nu_1"] & \\
A \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & & B \arrow[dl, "\nu_2"] \\
& C_2 &
\end{tikzcd}$$

PROOF. The existence of factorizations (μ, C, ν) and of the map η are elementary algebraic facts. We must only prove that any such C is k-reduced. But $C \simeq \nu(C) \subseteq B$, and B is an algebra of functions. So C is also an algebra of functions. □

The above lemma plays an important role in characterizing canonical realizations, analogous to the role of its linear variant, Zeiger's lemma (KALMAN, FALB, and ARBIB [1969, Chapter 10, Lemma (6.2)]), in linear system theory.

(1.6) REMARK. *Products of k-reduced algebras are k-reduced*. Indeed, assume that A_j is a subalgebra of k^{X_j} for each j in J. Then $\prod_J A_j$ is an algebra of functions on the disjoint union of the X_j. □

It follows from the definition of the tensor product $\otimes$ that for any two homomorphisms $\mu\colon A \to C$ and $\nu\colon B \to C$ there exists a unique homomorphism

(1.7) $\mu \otimes \nu\colon A \otimes B \to C\colon \sum a_i \otimes b_i \mapsto \sum \mu(a_i)\nu(b_i)$.

(1.8) LEMMA. *Let* A, B, C *be* *k-algebras. Then:*

(a) *The assignment* $(\mu, \nu) \to \mu \otimes \nu$ *establishes a bijection between* $\text{Hom}\,(A, C) \times \text{Hom}\,(B, C)$ *and* $\text{Hom}\,(A \otimes B, C)$.

(b) $X(A \otimes B)$ is naturally identified with $X(A) \times X(B)$.

(c) *If* A, B *are* *k-reduced, so is* $A \otimes B$.

PROOF. (a) Write $j_1\colon A \to A \otimes B\colon a \mapsto a \otimes 1$ and $j_2\colon B \to A \otimes B\colon b \mapsto 1 \otimes b$. We then define the inverse of the assignment $(\mu, \nu) \mapsto \mu \otimes \nu$ as $\text{Hom}\,(A \otimes B, C) \to \text{Hom}\,(A, C) \times$ $\times\ \text{Hom}\,(B, C)\colon \gamma \mapsto (\gamma\circ j_1, \gamma\circ j_2)$.

(b) Apply (a) to $C := k$.

(c) Since $X(A \otimes B)$ has been identified to $X(A) \times X(B)$, it is enough to prove the following: if $(\mu, \nu)(c) = 0$ for some c in $A \otimes B$ and for all (μ, ν) in $X(A) \times X(B)$, then $c = 0$. Assume that $c \neq 0$, and express c as a finite sum $\sum a_i \otimes b_i$ with the b_i linearly independent over k. For any μ in $X(A)$, consider the element $b := \sum \mu(a_i)b_i$ of B. For any ν in $X(B)$, $\nu(b) = \sum \mu(a_i)\nu(b_i) = (\mu, \nu)(c) = 0$. Since B is k-reduced, the function $b = 0$. The b_i being linearly independent, it follows that $\mu(a_i) = 0$ for all i. Since A is k-reduced and μ was arbitrary, $a_i = 0$ for all i. So $c = 0$. □

(1.9) REMARK. Recall that, in any category, the *coproduct* of a family $\{\Upsilon_\lambda, \lambda \in \Lambda\}$ of objects is defined as an object $\Upsilon = \coprod_\Lambda \Upsilon_\lambda$ and morphisms $\upsilon_\lambda\colon \Upsilon_\lambda \to \Upsilon$ which satisfy: for any object Θ and morphisms $\theta_\lambda\colon \Upsilon_\lambda \to \Theta$ there exists a unique morphism $\upsilon\colon \Upsilon \to \Theta$ such that $\upsilon\circ\upsilon_\lambda = \theta_\lambda$ for all λ. An equivalent way of expressing the properties (1.8.a) and (1.8.c) is to say that $A \otimes B$ (together with the inclusions

$a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$) is the coproduct of A and B in the category $\mathbf{Alg}_k^{\mathrm{red}}$ of k-reduced algebras, having (k-algebra) homomorphisms as its morphisms.

By induction, the coproduct of any *finite family* $A_1, \dots, A_n$ in $\mathbf{Alg}_k^{\mathrm{red}}$ is $A_1 \otimes \dots \otimes A_n$. *In the case of the category* $\mathbf{Alg}_k$ *of all* k-*algebras* (as in all *algebraic* categories), *an arbitrary* (not finite) *coproduct exists*. In fact, this coproduct can be obtained by the following direct-limit construction. Consider first the disjoint union of all $\{\otimes_{\Lambda_0} A_\lambda,\ \Lambda \supseteq \Lambda_0 = \text{finite}\}$. Then, for each pair $\Lambda_0 \subseteq \Lambda_0'$, identify $\otimes_{\Lambda_0} A_\lambda$ with the image of the inclusion morphism $\otimes_{\Lambda_0} A_\lambda \to \otimes_{\Lambda_0'} A_\lambda$ obtained by adding coordinates equal to 1 in the positions λ in $\Lambda_0' - \Lambda_0$. The algebra A obtained from this construction is the coproduct of the A_λ in $\mathbf{Alg}_k$. *If all* A_λ *are* k-*reduced, so is* A. Indeed, by the construction just sketched, an element x of A is in some (finite) tensor product $\otimes_{\Lambda_0} A_\lambda$, and every homomorphism $\otimes_{\Lambda_0} A_\lambda \to k$ extends to a homomorphism $A \to k$ (just define $A_\lambda \to k$ arbitrarily if λ is not in Λ_0). Assume now that x is in the kernel of all homomorphisms $A \to k$. Then x is in the kernel of all homomorphisms $\otimes_{\Lambda_0} A_\lambda \to k$, so by (2.8.c), $x = 0$. Therefore A is k-reduced, and *a fortriori* A is the coproduct of the A_λ in $\mathbf{Alg}_k^{\mathrm{red}}$. Thus *arbitrary coproducts exist in the category of* k-*reduced algebras*. Moreover, the construction shows that $A = \amalg A_\lambda$ includes all the A_λ, and that A is *generated by the* k-*algebras* A_λ. Finally, observe that the categorical definition of coproduct given above, applied to $\Theta := k$, shows that the set $X(A)$ of morphisms $A \to k$ is identified, through composition with the inclusion homomorphisms $A_\lambda \to A$, with the set $\prod_\Lambda X(A_\lambda)$ of families of homomorphisms $A_\lambda \to k$. □

(1.10) DEFINITION. *Let* A *be an algebra of functions on a set* X. *Take* x, y *in* X. *Then* A *separates* x *and* y *iff there exists an* a *in* A *such that* $a(x) \neq a(y)$.

(1.11) DEFINITION. *Let* A *be a subalgebra of the* k-*reduced algebra* B. *Then* A *is maximally separating with respect to* B *iff there is no* C *such that* $A \subsetneqq C \subseteq B$, *and* C *separates the same points of* $X(B)$ *as* A.

We shall later make use of the following

(1.12) EXAMPLE. Let $A := k[T_1T_2^n,\ n \geq 0]$ be the subalgebra of $k[T_1, T_2]$ generated by all the monomials $T_1T_2^n$, $n \geq 0$. Then A *is maximally separating with respect to* $B := k[T_1, T_2]$. Note that A identifies all points of the form $(0, x_2)$ and separates any other pair of points in $X(B) = k^2$. Assume that there is a C as in (1.11). Take $P(T_1, T_2) = \sum a_{ij}T_1^iT_2^j$ in C. Since C separates no more points than A, $P(0, x_2) = P(0, 0)$ for all x_2. Therefore $\sum a_{0j}T_2^j$ is a constant polynomial. Since k is infinite, P has no terms in T_2^j, $j > 0$. So $P = a_{00} + \sum_{i>0} a_{ij}T_1^{i-1}(T_1T_2^j)$ is in A. □

Recall (BOURBAKI [1972, Chapter V]) that the algebra A is *integral* over the subalgebra B iff every a in A satisfies an equation

$$a^n + b_1a^{n-1} + \ldots + b_n = 0,$$

for some b_j, $j = 1, \ldots, n$ in B and for some $n \geq 0$.

A k-algebra A is *finitely generated* iff there exists a finite subset $\{a_1, \ldots, a_s\}$ of A such that each element of A can be expressed as a finite combination of $a_1, \ldots, a_s$ using sums, products, and multiplication by elements of k.

(1.13) LEMMA. *Let the* k-*algebra* A *be an integral domain, and assume that* A *is integral over a subalgebra* B.

(a) *If* B *is a finitely generated* k-*algebra and* $Q(A)$ *is a finitely generated field extension of* $Q(B)$, *then* A *is a finite* B-*module and so also finitely generated as a* k-*algebra*.

(b) *If* A *is a finitely generated* k-*algebra then* B *is also finitely generated*.

PROOF. (a) This is an easy consequence of BOURBAKI [1972, V.3.2, Theorem 2].

(b) Let $a_1, \ldots, a_n$ generate A. Each a_i satisfies an integral equation with coefficients b_{ij} in B; call $C \subseteq B$ the k-algebra generated by all the b_{ij}. Then A is integral over C, and so B is integral over C. By (a), B is finitely generated. □

2. The Zariski Topology.

For the rest of this chapter, unless the contrary is explicitely stated, $A, B, C, \ldots$ will always denote k-reduced algebras.

Recall that $X(A)$ denotes the set of all homomorphisms $\mu: A \to k$.

We now introduce an operator V which assigns a subset $V(S)$ of $X(A)$ to each subset S of A. It is defined as

(2.1) $V(S) := \{x \text{ in } X(A) \mid a(x) = 0 \text{ for all } a \text{ in } S\}$.

Thus $V(S)$ is the set of solutions of the simultaneous equations $a(x) = 0$, a in S. Since, by convention $a(x)$ means $x(a)$, where $x: A \to k$ is a homomorphism, we can give the equivalent definition

(2.2) $V(S) := \{x: A \to k \mid \ker x \supseteq S\}$.

(2.3) PROPOSITION. The operator V satisfies the following properties:

(a) $V(S) = X(A)$ if and only if $S = \{0\}$.

(b) $V(A) = \emptyset$.

(c) $S \subseteq T$ implies $V(T) \subseteq V(S)$.

(d) Let $\langle S \rangle$ denote the ideal of A generated by S. Then $V(S) = V(\langle S \rangle)$.

(e) $V(\bigcup \{S_\lambda, \ \lambda \text{ in } \Lambda\}) = \bigcap \{V(S_\lambda), \ \lambda \text{ in } \Lambda\}$.

(f) Let $I.J$ denote the product of the ideals I, J of A, i.e. the ideal generated by $\{ab, \ a \text{ in } I, \ b \text{ in } J\}$. Then

$$V(I) \cup V(J) = V(I \cap J) = V(I.J).$$

PROOF. (a), (b), (c), (d), and (e) are easy consequences of the definition of V.

We now prove (f). By (c), $I \cap J \subseteq I$ and $I \cap J \subseteq J$ imply $V(I) \cup V(J) \subseteq V(I \cap J)$. Similarly, $I.J \subseteq I \cap J$ implies $V(I \cap J) \subseteq V(I.J)$. Consequently, it will be sufficient to show that if x is not in $V(I) \cup V(J)$ then x is not in $V(I.J)$. But if x belongs to neither $V(I)$ nor $V(J)$, there are a in I, b in J such that $a(x) \neq 0$, $b(x) \neq 0$. So $(a.b)(x) = a(x)b(x) \neq 0$. Since $ab \in I.J$, x is not in $V(I.J)$. □

It follows from (a), (b), (e), and (f) above that $\{V(S),\ S \subseteq A\}$ is the family of closed sets for a topology on $X(A)$, called the Zariski topology. Therefore we shall henceforth refer to sets of the type $V(S)$, $S \subseteq A$, as closed sets.

Occasionally it is convenient to define closed subsets of $X(A)$ in an indirect manner. Let A, B be any k-reduced algebras, and consider the tensor product $A \otimes B$. Let S be any subset of $A \otimes B$. For any $s = \sum a_i \otimes b_i$ in S and any x in $X(A)$ let $s(x)$ be the element $\sum a_i(x)b_i$ of B.

(2.4) LEMMA. $V^B(S) := \{x \in X(A) \mid s(x) = 0 \text{ for all } s \text{ in } S\}$ is a closed subset of $X(A)$.

PROOF. Let $\{b_\lambda,\ \lambda \text{ in } \Lambda\}$ be a basis for B as a k-vector space. Each s can be expressed in the form $\sum a_{\lambda,s} \otimes b_\lambda$ (finite sum), so $s(x) = \sum a_{\lambda,s}(x)b_\lambda$. Since $\{b_\lambda\}$ is linearly independent,

$$V^B(S) = V(\{a_{\lambda,s} \text{ in } A,\ s \text{ in } S,\ \lambda \text{ in } \Lambda\}).$$ □

We now define an operator I which associates a subset $I(Z)$ (in fact, an ideal) of A to each subset Z of $X(A)$. It is defined as

(2.5) $I(Z) := \{a \text{ in } A \mid a(x) = 0 \text{ for all } x \text{ in } Z\}$.

Thus $I(Z)$ is the annihilator of Z; it is evidently an ideal of A. We can define $I(Z)$ equivalently via homomorphisms, which gives

(2.6) $I(Z) := \bigcap\{\ker x,\ \ x \text{ in } Z\}$.

(2.7) PROPOSITION. *The operator* I *has the following properties:*

(a) $I(X(A)) = \{0\}$.

(b) $Z_1 \subseteq Z_2 \Rightarrow I(Z_2) \subseteq I(Z_1)$.

(c) $Z \subseteq V(I(Z))$, $S \subseteq I(V(S))$.

(d) $I\,(\bigcup\{Z_\lambda,\ \lambda \text{ in } \Lambda\}) = \bigcap\{I(V_\lambda),\ \lambda \text{ in } \Lambda\}$.

(e) $I(V(S))$ *is the intersection of all the* k-*ideals containing* S.

(f) $V(I(Z)) = \{x \text{ in } X(A) \mid \ker x \supseteq \bigcap_{z \in Z} \ker z\}$.

PROOF. (a), (b), (c), and (d) are easy consequences from (2.5), (2.6).

(e) Observe that, by (2.6),

$$I(V(S)) = \bigcap\{\ker x,\ \ x \text{ in } V(S)\},$$

and by (2.2),

$$x \text{ is in } V(S) \text{ iff } S \subseteq \ker x.$$

(f) Similar to (e). □

(2.8) REMARK. Using (2.3) and (2.7) it is easy to verify the following facts:

(i) $I(V(I(Z))) = I(Z)$,

(ii) $V(I(V(Z))) = V(Z)$,

(iii) the Zariski closure of any $Z \subseteq X(A)$ is $V(I(Z))$, and

(iv) an ideal I is of the form $I(Z)$ if and only if $I = I(V(I))$.

Ideals as in (iv) are called *closed*; this terminology comes from regarding $S \mapsto I(V(S))$ as an *algebraic* (not *topological*) *closure operator*.

One could have deduced (i)-(iv) from the fact that, by (2.3c), (2.7b) and (2.7c), the pair $\{V, I\}$ constitutes a *duality* or *Galois connection* (KUROSH [1963, § 51]). Finally, it also follows that $\{V, I\}$ *establish an inclusion-reversing bijection between closed ideals of* A *and closed subsets of* $X(A)$. □

(2.9) LEMMA. *For any ideal* I *of* A, $X(A/I)$ *can be naturally identified with* $V(I)$; *therefore the canonical map* $\pi\colon A \to A/I$ *can be naturally identified with the restriction map* $a \mapsto a|V(I)$, *where* a *is viewed as a function on* $X(A)$. *Furthermore,* I *is closed if and only if* A/I *is* k-*reduced*.

PROOF. From elementary algebraic considerations, the injection $X(A/I) \to X(A)\colon y \mapsto y\circ\pi$ shows that there is a bijection between functions y in $X(A/I)$ and those functions $x = y\circ\pi$ in $X(A)$ for which $x|I = 0$, that is, $I \subseteq \ker x$. These are precisely the x in $V(I)$.

To prove the second part, note that k-ideals of A/I correspond via π to those k-ideals of A which contain I. Thus $\mathrm{rad}_k\, A/I = 0$ precisely when I is the intersection of all the k-ideals containing it. Applying (2.7e) to (2.8iv) gives the proof. □

We need the following elementary topological

(2.10) DEFINITION. *A topological space* Z *is irreducible iff* Z *is not the union of two proper closed subsets, in other words, iff*

$$Z = Z_1 \cup Z_2,\quad Z_i\ \ \text{closed, implies}\ \ Z = Z_1\ \ \text{or}\ \ Z = Z_2.$$

Clearly, Z is irreducible iff any two nonempty open sets of Z have a nonempty intersection, i.e. iff *any open subset of* Z *is dense*. Therefore, *irreducibility permits the use of local intuition and methods* (= arguments about neighborhoods) *in the proof of global statements*.

To apply the above concepts in our context, we must study more closely the Zariski topology in the spaces $X(A)$. This topology will in general not be Hausdorff (= different points having disjoint neighborhoods), but it is true that *each point of* $X(A)$ *is a closed set*. Indeed, by (2.7f), $\overline{\{x\}} = \{z \mid \ker z \supseteq \ker x\}$; since $\ker x$ is a maximal ideal, it follows that $\overline{\{x\}} = \{x\}$ as wanted.

A set $Z \subseteq X(A)$ can be given the subspace topology; thus irreducibility of Z is well defined. In particular, it is easy to see that a closed subset V of $X(A)$ is irreducible iff

(2.11) $V \subseteq V_1 \cup V_2$, V_i both closed in $X(A)$, implies $V \subseteq V_1$ or $V \subseteq V_2$.

Recall that an ideal P of A, $P \neq A$, is *prime* iff, for any ideals J_1, J_2, $J_1 . J_2 \subseteq P$ implies $J_1 \subseteq P$ or $J_2 \subseteq P$. Equivalently, P is prime iff, for any a, b in A, ab in P implies either a is in P or b is in P; see BOURBAKI [1972, II.1.1]. We then have:

(2.12) LEMMA. *Let* V *be a closed subset of* $X(A)$. *Then* V *is irreducible if and only if* $I(V)$ *is prime*.

PROOF. ["only if"] Assume $J_1 . J_2 \subseteq I(V)$. Then, by (2.3c), $V = V(I(V)) \subseteq V(J_1) \cup V(J_2)$. Since V is irreducible, $V \subseteq V(J_1)$ or $V \subseteq V(J_2)$. Thus by (2.6c,b) $J_1 \subseteq I(V(J_1)) \subseteq I(V)$ or $J_2 \subseteq I(V(J_2)) \subseteq I(V)$.

["if"] Assume $V = V_1 \cup V_2$, V_i closed. Then by (2.7d) $I(V_1) . I(V_2) \subseteq I(V_1) \cap I(V_2) = I\ (V_1 \cup V_2) = I(V)$. This and $I(V)$ = prime imply $I(V_1) \subseteq I(V)$ or $I(V_2) \subseteq I(V)$. Thus $V = VI(V) \subseteq VI(V_1) = V_1$ or $V \subseteq V_2$. □

In particular, $X(A)$ is irreducible iff A is an integral domain.

It follows from (2.12) that $V \mapsto I(V)$ and $I \mapsto V(I)$ establish an inclusion-reversing bijection between irreducible subsets of $X(A)$ and prime ideals of A.

3. k-*Spaces*.

A k-space is the abstract version of a space of the type $X(A)$:

(3.1) DEFINITION. *A k-space* $(X, A(X))$, *or simply* X, *is a set* X *and a k-algebra* $A(X)$ *of functions* $X \to k$ *such that the map* $X \to X(A(X))$: $x \mapsto e_x$ (= *evaluation at* x) *is bijective. The elements of* $A(X)$ *are the polynomial functions on* X.

The terminology "polynomial functions" is motivated by Example (1.4).

We always consider a k-space X as a topological space, with topology induced from the Zariski topology on $X(A)$ by the bijection $X \simeq X(A(X))$. We shall see presently that all k-spaces are essentially of the type $(X(A), \iota(A))$, where A is k-reduced an ι is the map introduced in (1.2).

(3.2) DEFINITION. *Let* X, Y *be k-spaces. A map* $f: X \to Y$ *is polynomial iff for each polynomial function* $b: Y \to k$ *in* $A(Y)$ *the composition* $b \circ f: X \to k$ *is a polynomial function in* $A(X)$.

We shall use X, Y, Z to indicate k-spaces; $f: X \to Y$ will always mean a polynomial map.

It can be trivially verified that k-spaces as objects, together with polynomial maps as morphisms, constitute a category. In this category we have the following

(3.3) LEMMA. *Every k-space is isomorphic to a k-space of the type* $(X(A), \iota(A))$, *where* A *is k-reduced and* ι *is the map introduced in* (1.2).

SKETCH OF PROOF. Given a k-space $(X, A(X))$, let $A := A(X)$. An easy check shows that $X \to X(A)$: $x \mapsto e_x$ is the required isomorphism of k-spaces. □

Since all results will be stated up to isomorphism, we are justified by (3.3) when carrying out our proofs about k-spaces only for those of type $(X(A), \iota(A))$. We continue to use our convention of identifying $\iota(A)$ with A (so $A(X(A)) = A$) when there is no danger of confusion.

A major rôle is played by the k-spaces introduced by the following

(3.4) DEFINITION. *A k-space X is a variety iff A(X) is a finitely generated k-algebra.*

Let $a_1, \ldots, a_n$ generate the k-algebra A. Generation is equivalent to ontoness of the k-algebra homomorphism $\mu\colon k[T_1, \ldots, T_n] \to A$ defined by the assignment $T_i \mapsto a_i$. By (1.4) and (2.9) we see that $(X(A), \iota(A))$ is isomorphic to (V, B), where V is the set of points x in k^n which satisfy all equations $f(x) = 0$, f in $\ker \mu$, and where the functions in B are the restrictions to V of the polynomial functions $k^n \to k$. There are many possible representations (V, B) for each A, depending upon the choice of generators $a_1, \ldots, a_n$. Thus, (affine) *varieties are the coordinate-free versions of those subsets of k^n defined by polynomial equations*. When studying varieties, it is usually simpler to deal with representations of the type (V, B). Note that if (V_1, B_1) and (V_2, B_2) are (concrete) varieties, where $V_1 \subseteq k^n$ and $V_2 \subseteq k^m$, a map $f\colon V_1 \to V_2$ is polynomial precisely when f is defined by an m-vector of n-variable polynomials.

(3.5) DIGRESSION. Let $k = \mathbb{R}$, the reals. Then varieties have a natural topology induced from their embedding in Euclidean space with the usual topology. This topology is finer than the Zariski topology; (for instance the only proper Zariski-closed subsets of R^1 are finite sets). More generally, given any normed field k, we may define a *strong topology* on k-spaces by choosing as a basis of open sets all finite intersections of sets of the type $f^{-1}(N)$, for all polynomial functions f and all open sets N in k, i.e. the coarsest topology in X for which all f in A(X) become continuous for the *normed* topology of k. (See for instance SHAFEREVICH [1975, Chapter 7] for the case of varieties over $\mathbb{C}$.) For the purpose of this work, realization theory, we are mainly interested in (global) questions of structure; thus we shall use only the Zariski topology, even in the cases $k = \mathbb{R}$ or $\mathbb{C}$, except for the proof of some technical facts on varieties over $\mathbb{R}$ in Section 4. □

(3.6) DEFINITION. *Let* $g: X_1 \to X_2$. *The transpose of* g *is the homomorphism* $A(g): A(X_2) \to A(X_1)$ *defined by*

$$A(g)(f) := f \circ g.$$

(3.7) LEMMA. *Fix two* k-*spaces* X_1, X_2. *Then the assignment* $A: g \mapsto A(g)$ *establishes a bijection between polynomial maps* $X_1 \to X_2$ *and* k-*algebra homomorphisms* $A(X_2) \to A(X_1)$.

We shall write $X(\mu): X(B) \to X(A)$ for the polynomial map corresponding to the homomorphism $\mu: A \to B$.

PROOF. The problem is to define an inverse X of the transpose functor A. Let the k-spaces X_j be $(X(A_j), \iota(A_j))$, $j = 1, 2$ and $\iota_j: A_j \to k^{X_j}$ the canonical maps. Let $\alpha: \iota_2(A_2) \to \iota_1(A_1)$ and take x in X. Since $e_x: \iota_1(A_1) \to k$ is a homomorphism, $e_x \circ \alpha \circ \iota_2: A_2 \to k$ is also a homomorphism. Now define

$$X(\alpha): X(A_1) \to X(A_2): \alpha \to e_x \circ \alpha \circ \iota_2.$$

It is easy to verify that $X(A(g)) = g$ and $A(X(\alpha)) = \alpha$ for all $g: X(A_1) \to X(A_2)$ and all $\alpha: \iota_2(A_2) \to \iota_1(A_1)$. □

(3.8) COROLLARY. *The category of* k-*spaces is dual (arrows reversed) to the category of* k-*reduced* k-*algebras*. □

The above duality allows the translation of constructions and statements about algebras into (dual) statements about k-spaces, and vice versa. For instance, (1.8) says that the categorical product $X \times Y$ of two k-spaces X, Y is the k-space $X(A(X) \otimes A(Y))$ and that the underlying set of this k-space is the cartesian product $X \times Y$. By induction, $X(A_1) \times X(A_2) \times \ldots \times X(A_n) = X(A_1 \otimes \ldots \otimes A_n)$. And, in particular, $(X(k[T]))^n$ (n-th fold power) coincides with $X(k[T_1, \ldots, T_n]) = k^n$; see Example (1.4). This also shows that the notation k^n is consistent with products in the category of k-spaces. (Note that, in particular, $k^0 = X(k)$ = one point, say $\{0\}$).

As an example of the transpose construction, consider a function $f: X \to k$. Since the transpose $A(f): k[T] \to A(X)$ is a k-algebra homomorphism, $A(f)$ is determined by $A(f)(T) = T \circ f$. Since T is the identity map on k,

(3.9) $A(f)(T) = f$.

So the transpose of f is the map $P(T) \mapsto P(f)$.

We now relate various properties of polynomial maps to properties of their transposes.

(3.10) DEFINITION. *A polynomial map* $f: X \to Y$ *is dominating iff* $\overline{f(X)} = Y$; f *is a closed embedding iff* f *can be factored as* $g_1 \circ g_2$, *where* g_2 *is an isomorphism* $X \simeq V$ *and* g_1 *is the inclusion map* $V \to Y$, *for some closed subset* V *of* Y.

(3.11) LEMMA. *Let* $\alpha: A \to B$ *and denote* $f := X(\alpha): X(B) \to X(A)$. *Then*

(a) $f^{-1}(V(S)) = V(\alpha(S))$ *for any* $S \subseteq A$.

(b) f *is continuous*.

(c) $\overline{f(V(I))} = V(\alpha^{-1}(I))$ *for any closed ideal* I *of* B.

(d) f *is dominating if and only if* α *is one-to-one*.

(e) f *is a closed embedding if and only if* α *is onto*.

PROOF. (a) x is in $f^{-1}(V(S))$
iff $f(x)$ is in $V(S)$,
iff $a(f(x)) = 0$ for all a in S,
iff $\alpha(a)(x) = 0$ for all a in S,
iff x is in $V(\alpha(S))$.

(b) All closed sets in $X(A)$ are by definition of the form $V(S)$. By (a), pre-images of closed sets are closed.

(c) We first prove that $\alpha^{-1}(I) = I(f(V(I)))$; in fact the following statements are equivalent:

a is in $I(f(V(I)))$,

$a(f(x)) = 0$ for all x in $V(I)$,

$\alpha(a)$ is in $I(V(I)) = I$ (I = closed!),

a belongs to $\alpha^{-1}(I)$.

Therefore $V(\alpha^{-1}(I)) = V(I(f(V(I)))) = \overline{f(V(I))}$, as required.

(d) Applying (c) to $I = \{0\}$, $V(\alpha^{-1}(I)) = \overline{f(X)}$. So f is dominating iff $V(\ker \alpha) = Y$, which by (1.11a) is equivalent to $\ker \alpha = \{0\}$.

(e) Dualizing (3.10), f is a closed embedding iff the transpose homomorphism α factors as $\beta_2 \circ \beta_1$, where β_1 is a homomorphism $B \to B/I$ for some ideal I and β_2 is an isomorphism. Such factorizations exist precisely when α is onto. □

Both dominating maps and closed embeddings will play important roles in our treatment of realization theory. We emphasize some intuitive aspects of these concepts through the following

(3.12) DISCUSSION. It follows from (e) above that α onto implies that f is one-to-one. The converse is false. For an easy example, consider $X = Y = k := \underline{\underline{R}}$, $f_1(x) := x^3$. Then, f_1 is one-to-one, but $A(f_1): k[T] \to k[T]: T \mapsto T^3$ is not onto (T is not in the image). The problem does not lie in the fact that $\underline{\underline{R}}$ is not algebraically closed: for any field k we may consider $f_2: k \to k^2: x \mapsto (x^2, x^3)$; f_2 is one-to-one but $A(f_2): k[T_1, T_2] \to k[T]: T_1 \mapsto T^2$, $T_2 \mapsto T^3$ has image $k[T^2, T^3] \neq k[T]$.

A variation of the last example provides a bijective map f_3 which is not an isomorphism. Indeed, consider the "cusp"

$$Y := \{(x, y) \text{ in } k^2 | x^3 = y^2\}.$$

Then

(3.13) $f_3: k \to Y: x \mapsto (x^2, x^3)$, is bijective. But f is not an isomorphism, because, by the equivalence of categories (1.7), f_3 is an isomorphism iff $A(f_3)$ is an isomorphism. But $A(Y) \simeq k[T^2, T^3] \not\simeq k[T]$; see DIEUDONNE [1974, 1.1, Example 5]. Intuitively, we cannot expect to have any isomorphism between k and Y because the curve Y has a singularity (at the origin) while the line k has none.

It is not difficult to prove that, in the category of k-spaces, monomorphism = one-to-one and epimorphism = dominating. A dominating map is in general not onto. This is illustrated over $k = \mathbb{R}$ by $X = Y := \mathbb{R}$, $f(x) := x^2$; this is a dominating map because the smallest Zariski closed set containing the nonnegative reals is all of $\mathbb{R}$. In the particular case when k is algebraically closed and X is an irreducible variety, a dominating f becomes almost onto, in the sense that $f(X)$ contains a Zariski open subset of Y; see (3.14) below. Thus in this particular case $f(X)$ is all of Y except at most for a subset of "lower dimension" (to be made precise later). Moreover, it can be proved that, when $k = \mathbb{C}$ and Y is given the strong topology (3.4), $f(X)$ has a nowhere dense complement. □

We remarked above that the image of a polynomial map is in general not a closed set. When $f: X \to Y$ is a polynomial map between two varieties, one can sometimes characterize $f(X)$ as a set defined by polynomial equalities *and* inequalities. A *constructible* subset C of a variety X is a finite union of sets of the type $U \cap F$, where U is open and F is closed. In other words, C is in the Boolean algebra generated by the Zariski topology of X. When k is a real-closed field (JACOBSON [1964, VI.2]), like $k = \mathbb{R}$, we define an *real-constructible* set C as a finite union of sets $V \cap F$, where F is closed and U is of the type $\{x \text{ in } X \mid f(x) < 0\}$ for some polynomial function f.

(3.14) THEOREM. *Let* X, Y *be varieties and let* $f: X \to Y$. *Then*

(a) *If* k *is algebraically closed and* C *is a constructible subset of* X (*e.g.* $C = X$), *then* $f(X)$ *is a constructible subset of* Y.

(b) *If* k *is a real-closed field (e.g.* $k = \mathbb{R}$) *and* C *is a real-constructible subset of* X (*e.g.* $C = X$), *then* $f(X)$ *is a real-constructible subset of* Y.

(c) *If* f *is dominating,* Y *is irreducible, and* k *is algebraically closed*, $f(X)$ *contains a (Zariski) open subset of* Y.

(d) *If* f *is dominating,* Y *is irreducible and* $k = \mathbb{R}$, *then* $f(X)$ *contains a set open in the strong topology of* Y.

PROOF. (a) This is the well-known CHEVALLEY's theorem; see for instance DIEUDONNE [1974, Chapter 4, Corollary to Proposition 14].

(b) This statement is essentially the *generalized* STURM's *Theorem* due to TARSKI and SEIDENBERG; see JACOBSON [1964, VI.10]. (The usual statement of the TARSKI-SEIDENBERG result requires that the coefficients of f be rational, so that $f(C)$ can be constructed algorithmically. However, the proof itself does not depend upon this requirement; see SEIDENBERG [1954, footnote in page 366].)

To prove (c) and (d), write $f(X) = \bigcup_{\text{finite}} U_i \cap F_i$ as in the definition of constructibles and real-constructibles, with the F_i closed and the U_i Zariski-open or, when $k = \mathbb{R}$, of the type $\{f(x) < 0\}$, so open in the strong topology. If $F_i \neq Y$ for all i, then, by irreducibility of Y, $\overline{f(X)} = \bigcup F_i \neq Y$, contradicting domination of f. So some $F_i = Y$, and $f(X)$ contains U_i. ☐

An important type of dominating map arises in the following

(3.15) DEFINITION. *The principal open set defined by* $a \in A$ *is*

$$D(a) := \{x \text{ in } X(A) \mid a(x) \neq 0\}.$$

The principal open sets constitute a basis for the Zariski topology. Indeed, the complement of any closed set $V(S)$ is the union of the $D(a)$, a in S. For simplicity, let A be an integral domain. Denote by $a^{-1}A$ the algebra $A \subseteq a^{-1}A \subseteq Q(A)$ consisting of all b/a^n, b in A, $n \geq 0$. Take any $\alpha\colon a^{-1}A \to k$. Then $\beta := \alpha|A\colon A \to k$ satisfies $\beta(a) \neq 0$. Conversely, if $\beta\colon A \to k$ and $\beta(a) \neq 0$ then the rule

$\alpha(b/a^n) := \beta(b)/\beta(a)^n$ defines the (unique) α extending β. Therefore $D(a)$ is the image of the map

(3.16) $X(a^{-1}A) \to X(A)$,

dual to the inclusion $A \subseteq a^{-1}A$. This map is both one-to-one and dominating and establishes a homeomorphism between $X(a^{-1}A)$ and $D(a)$; see BOURBAKI [1972, II.4.3, Corollary to Proposition 13]. Local arguments about k-spaces are often simplified by restricting attention to principal open sets. ☐

We shall be especially interested in "quotients" of varieties:

(3.17) DEFINITION. A k-space X is an almost-variety iff there exists a variety $\hat{X}$ and a dominating polynomial map $f: \hat{X} \to X$.

By definition of "dominating", this means that X has a dense subset $f(\hat{X})$ consisting of equivalence classes of elements of X. Let $f: \hat{X} \to X$ as above. By Hilbert's basis theorem we may write $\hat{X} = \hat{X}_1 \cup \ldots \cup \hat{X}_r$, where the $\hat{X}_i$ are irreducible closed sets; see BOURBAKI [1972, III.2.10, Corollary 3 of Theorem 3]. From the definition of irreducibility, it is easy to prove that continuous images and closure of irreducible sets are irreducible. Hence $X_i := \overline{f(\hat{X}_i)}$ is an irreducible closed subset for each i. Since $f|\hat{X}_i: \hat{X}_i \to X_i$ is dominating and each $\hat{X}_i$ is a variety, X can be written as a finite union of irreducible almost-varieties.

The next lemma shows that every irreducible almost-variety has an open (hence dense) subset which is a variety, justifying the terminology "almost variety".

(3.18) LEMMA. Let X be an irreducible almost-variety. Then there exists a principal open set $D(a)$ of X which is a variety, i.e. such that $a^{-1}A(X)$ is finitely generated.

PROOF. Let $f: \hat{X} \to X$ be dominating with $\hat{X}$ an irreducible variety. Then $A(f): A(X) \to A(\hat{X})$ embedds $A(X)$ in $B := A(\hat{X})$; we identify $A(X)$ with its image $A \subseteq B$. By BOURBAKI [1972, V.3.1., Corollary 1] there is an a in A and $T_1, \ldots, T_r$ in B such that the finitely generated algebra $a^{-1}B$ is integral over the polynomial ring $a^{-1}A[T_1, \ldots, T_r]$. By (1.13) $a^{-1}A[T_1, \ldots, T_r]$ is a finitely generated algebra. Let $P_1(T_1, \ldots, T_r), \ldots, P_m(T_1, \ldots, T_r)$ generate $a^{-1}A[T_1, \ldots, T_r]$. Then $P_1(0, \ldots, 0), \ldots, P_m(0, \ldots, 0)$ generate $a^{-1}A$. Thus $a^{-1}A$ is finitely generated □

We shall find in Chapter IV that the canonical state-spaces are in general almost-varieties. In particular we shall give an input/output map whose canonical state-space is that given by the following

(3.19) EXAMPLE. Let $A := k[T_1T_2^n,\ n \geq 0]$. Since $A \subseteq k[T_1, T_2]$, $X(A)$ is an almost-variety. Take $a := T_1$. Then $a^{-1}A = k[T_1, T_2, T_1^{-1}]$ is a finitely generated algebra. As a variety, $X(a^{-1}A)$ can be represented by the set of solutions (x, y, z) in k^3 of the equation $xz - 1 = 0$. Note that $X(a^{-1}A)$ can be also naturally viewed as the principal open set $x_1 \neq 0$ in k^2. A point x of $X(A)$ not in $D(a)$ satisfies $x(T_1) = T_1(x) = 0$ and (as we now show) it also satisfies $x(T_1T_2^n) = 0$ for all n. This is not trivial, since the "proof" $x(T_1T_2) = x(T_1).x(T_2) = 0.x(T_2) = 0$ is fallacious: $x(T_2)$ is not defined, because T_2 is not in A. One way to prove the statement is by means of the theorem of extension of places (see BOURBAKI [1972, VI.2.4, Proposition 13]). This theorem implies that, if there is no extension of x such that $x(T_2)$ is defined, then there does exist an overring of A containing T_2^{-1} and such that $x(T_2^{-1}) = 0$ (the values of the extension of x, though, are not necessarily in k). But if this is the case, then $x(T_1T_2^n) = x(T_1T_2^{n+1})x(T_2^{-1}) = 0$ for all n, as wanted. Therefore the complement of $D(a)$ consists of just one point and $X(A)$ is the disjoint union of the variety $X(a^{-1}A)$ and this one extra point. □

4. Dimension.

There are various possible notions of dimension for a k-space X. All these notions coincide if X is a variety and k is algebraically closed, but this is not true for more general X or k. Thus our choice will be to some extent arbitrary, to be justified by the results.

Let A be an overring of B having no zero divisors. Recall that elements $\ell_1, \ldots, \ell_n$ of A are algebraically dependent over B iff there exists a nonzero polynomial P in $B[T_1, \ldots, T_n]$ such that $P(\ell_1, \ldots, \ell_n) = 0$. When no such P exists, $\ell_1, \ldots, \ell_n$ are algebraically independent. We review some elementary properties of algebraic dependence. They can be found, for instance, in ZARISKI and SAMUEL [1958, II.12].

An arbitrary subset L of A is algebraically independent iff every finite subset of L is. A transcendence basis for A (over B) is a maximal algebraically independent set $L \subseteq A$; in other words, if s is not in L, then $L \cup \{s\}$ is algebraically dependent. All transcendence bases have the same cardinality, the transcendence degree $\text{trdeg}_B A$ of A over B. When B = k, we denote $\text{trdeg}_k A$ just by trdeg A. When $\text{trdeg}_B A = 0$, A is algebraic over B.

The notion of transcendence degree, which is based on "dependence", is analogous to that of dimension of vector spaces.

A transcendence basis for A over B can be extracted out of any system of generators of A over B. In particular, when A is a finitely generated B-algebra, $\text{trdeg}_B A$ is finite. If both A, B are k-algebras, then

(4.1) $\text{trdeg}\, A = \text{trdeg}\, B + \text{trdeg}_B A.$

(4.2) LEMMA. (ZARISKI and SAMUEL [1958, II.2, Theorems 28 and 29]). Let A, B be integral domains, and let $\varphi: A \to B$ be onto. Then $\text{trdeg}\, B \leq \text{trdeg}\, A$; if both are finite then equality can only hold when φ is an isomorphism. □

When the k-algebra A is not an integral domain, one can still define trdeg A, making use of the integral domains {A/P, P prime ideal of A}; just let

(4.3) $\quad \operatorname{trdeg} A := \sup\{\operatorname{trdeg} A/P,\ P \text{ prime ideal of } A\}$.

By (4.2), this definition is consistent with the basic definition for integral domains.

We define a notion of *dimension* for a k-space by setting

(4.4) $\quad \dim X := \operatorname{trdeg} A(X)$.

Observe that an *almost-variety* X *is always finite-dimensional*, since then $A(X)$ is included in a finitely-generated algebra.

A combinatorial consequence of the definition of dimension is:

(4.5) LEMMA. *Let* $\dim X = n$ *be finite. Let*

$$V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_s,$$

be a chain of irreducible closed subsets of X. *Then* $s \leq n$.

PROOF. Clear by (2.12) and induction on (4.2). □

Let $f: X \to Y$ be a polynomial map. Since f is continuous and points of Y are closed, the *fibers* $f^{-1}(y)$, y in Y, are closed subsets of X. Therefore each fiber is a k-space and as such has a well-defined dimension.

We shall see below how to generalize the dimension formulas for linear maps $(\dim X - \dim \ker f = \dim f(X))$ to the polynomial context. This will follow from a general result on the structure of polynomial maps.

The next theorem summarizes a number of relevant results in a form convenient for our purposes.

(4.6) THEOREM. *Let* X, Y *be two irreducible almost-varieties, with* $\dim X = n$, $\dim Y = m$. *Let* $f: X \to Y$ *be a dominating polynomial map. Then there exists an integer* $s \geq 0$, *irreducible varieties* X_1, Y_1 *and polynomial maps* $j_X: X_1 \to X$, $j_Y: Y_1 \to Y$, $f_1: X_1 \to Y_1$ *and* $g: X_1 \to Y_1 \times k^{n-m}$ *such that*

(a) j_X *[respectively* j_Y*] identifies* X_1 *[respectively* Y_1*] with a principal open set of* X *[respectively* Y*] as in (3.16).*

(b) *The following diagram commutes:*

$$\begin{tikzcd}
X \arrow[dd, "f"'] & X_1 \arrow[l] \arrow[dd, "f_1"] \arrow[dr, "g"] & \\
 & & Y_1 \times k^{n-m}, \arrow[dl, "pr_1"] \\
Y & Y_1 \arrow[l] &
\end{tikzcd}$$

where pr_1 *denotes the projection in the first factor.*

(c) g *is dominating and the sets* $g^{-1}(z)$ *have cardinality at most* s *for each* z *in* $Y_1 \times k^{n-m}$.

(d) *If* k *is algebraically closed, then* g *is onto and each* $g^{-1}(z)$ *has exactly* s *elements.*

(e) *If* $k = \mathbb{R}$ *then* X_1, Y_1 *are differentiable manifolds and the differentials* dg *and* df_1 *have full rank at every point.*

(f) *If either* $k = \mathbb{R}$ *or* k *is algebraically closed,*

$$\dim f_1^{-1}(f_1(x)) = n - m,$$

for each x *in* X_1.

PROOF. Let $A := A(Y)$ be identified through $A(f)$ to a subalgebra of $B := A(X)$. By (3.18), there exist a in A and b in B such that $a^{-1}A$ and $b^{-1}B$ are finitely generated. Then $a^{-1}A \subseteq a^{-1}B \subseteq a^{-1}b^{-1}B$. By BOURBAKI [1972, V.1.5], there is some s in $a^{-1}A$ such that $s^{-1}(a^{-1}A)$ is integrally closed in its quotient field. Replacing if necessary a by sa, we may assume that $a^{-1}A$ is integrally closed. By BOURBAKI [1972, V.3.1, Corollary 1], there exist $T_1, \ldots, T_{n-m}$ algebraically independent elements of $(ab)^{-1}B$ such that $(ab)^{-1}B$ is integral over $A' := a^{-1}A[T_1, \ldots, T_{n-m}]$. Since $a^{-1}A$ is integrally closed, A' is also integrally closed; see BOURBAKI [1972, V.1.3, Corollary 2].

Let s be the separable degree of the algebraic field extension $Q(B):Q(A')$. By BOURBAKI [1972, V.2.3, Remark 3], the fibers of the canonical map $X((ab)^{-1}B) \to X(A_1)$ have at most s elements. Let $X_1 := X((ab)^{-1}B)$, $Y_1 := X(a^{-1}A)$ and let j_X, j_Y, f_1, g be respectively the maps dual to the inclusions $B \subseteq (ab)^{-1}B$, $A \subseteq a^{-1}A$, $a^{-1}A \subseteq (ab)^{-1}B$, $A' \subseteq (ab)^{-1}B$. Then (a), (b), and (c) hold by construction.

If k is algebraically closed, there exists an open set V in $Y_1 \times k^{n-m}$ such that $g^{-1}(z)$ has precisely s elements for each z in V; see DIEUDONNE [1974, Chapter 5, Proposition 6]. Let $U \subseteq g^{-1}(V)$ be a principal open set. By (3.14), $f_1(U)$ contains a principal open set W. Let $\hat{U} \subseteq f_1^{-1}(W) \cap U$ be a principal open set. Then $g(\hat{U}) \subseteq V$ implies that, for any z in $Y_1 \times k^{n-m}$, either $g^{-1}(z) \cap \hat{U}$ is empty or it has exactly s elements. Replacing X_1 by $\hat{U}$ and Y_1 by W gives (d); g is surjective by DIEUDONNE [1972, IV.4, Corollary 1].

Let $k = \underline{\underline{R}}$. We replace X_1, Y_1 by nonsingular principal open subsets; see, for example, BRÖCKER [1975, Theorem 12.12]. We may further replace X_1 by a principal open set in which dg has maximal rank p. By (3.14), $g(X_1)$ contains a strong open set of $Y_1 \times k^{n-m}$, so by SARDS' theorem (see BRÖCKER [1975, Theorem 2.11]), $p = n$. Similarly with f_1. Thus (e) follows.

(f) For k algebraically closed, see DIEUDONNE [1974, Chapter 4, Theorem 2]; for $k = \underline{\underline{R}}$ this follows from (e) and BRÖCKER [1975, Theorem 1.9].

CHAPTER III. REALIZATION THEORY

We investigate in this chapter the general realization theory of an m-input, p-output polynomial response. Before doing this, we develop the formalism of Volterra series and prove some simple facts to be used later in the study of finiteness conditions.

Throughout this work, k denotes an infinite field; all parameters belong to k. The assumption that k is infinite is merely a technical convenience, permitting the identification of polynomials and polynomial functions.

Both m and p (number of input and output channels, respectively) are positive integers, arbitrary but fixed throughout the discussion.

5. Volterra Series.

We shall use the following notations:

$\underline{\underline{N}}$:= set of nonnegative integers.

$\underline{\underline{N}}^m$:= set of column m-vectors over $\underline{\underline{N}}$.

$\underline{\underline{N}}^{m \times t}$:= set of $m \times t$ matrices over $\underline{\underline{N}}$.

$(\underline{\underline{N}}^m)^*$:= set of all finite sequences of elements of $\underline{\underline{N}}^m$, including the empty sequence denoted by Λ.

If α is in $(\underline{\underline{N}}^m)^*$, $\alpha = \alpha_1 \ldots \alpha_t \neq \Lambda$ then $|\alpha|$:= length of $\alpha := t$; $\|\alpha\|$:= weight of $\alpha := \alpha_1 + \ldots + \alpha_t$; $|\Lambda| = \|\Lambda\| := 0$; α_{ij} := element in i-th row of column vector α_j, $j = 1, \ldots, t$, $i = i, \ldots, m$.

Δ := set of proper sequences: α in $(\underline{\underline{N}}^m)^*$ belongs to Δ iff $\alpha = \Lambda$ or $\alpha = \alpha_1 \ldots \alpha_t$ with $\alpha_t \neq 0$.

If $\alpha = \alpha_1 \ldots \alpha_t$ and $\beta = \beta_1 \ldots \beta_s$ are in $(N^m)^*$, $\alpha\beta$:= concatenation of α and $\beta := \alpha_1 \ldots \alpha_t \beta_1 \ldots \beta_s$.

If, say, $t \leq s$, then $\alpha + \beta$:= sum of α and $\beta := \gamma_1 \ldots \gamma_s$, where $\gamma_i := \alpha_i + \beta_i$ for $i = 1, \ldots, t$ (addition of α_i, β_i is row-wise in $\underline{\underline{N}}^m$) and $\gamma_i := \beta_i$ if $i = t + 1, \ldots, s$. Similarly if $t > s$.

Since an $\alpha = \alpha_1 \ldots \alpha_t$ in $(\underline{\underline{N}}^m)^*$ is a sequence of columns, we may regard α as an $m \times t$ matrix. Thus we may, and shall, make the following identification:

$$(\underline{\underline{N}}^m)^* = \bigcup_{t \geq 0} \underline{\underline{N}}^{m \times t}.$$

Under this identification, concatenation of α and β is the same thing as formation of the block matrix $[\alpha \vdots \beta]$; addition is simply addition of matrices (augmented by zeroes to the right if necessary). Observe that the notation α_{ij} is consistent with the matrix interpretation.

Each of the operations, concatenation and addition, make $(\underline{\underline{N}}^m)^*$ into a monoid; in both cases Λ is the identity. In both cases Δ is a submonoid, i.e. if α and β are both in Δ, then both $\alpha\beta$ and $\alpha + \beta$ are in Δ. We shall denote by $(\Delta, \cdot)$ and $(\Delta, +)$ the two monoids thus obtained. Both monoids will play a central role in our theory. The monoid $(\Delta, +)$ is used in defining "polynomial" and $(\Delta, \cdot)$ is used in obtaining finiteness conditions.

Let ξ_{ij}, $i = 1, \ldots, m$, $j = 1, 2, 3, \ldots$ denote denumerably many (distinct) indeterminates, and for each j let ξ_j denote the subset $\xi_{1j}, \ldots, \xi_{mj}$. Let $\alpha = \alpha_1 \ldots \alpha_t$ be in $(\underline{\underline{N}}^m)^*$ and define

$$\xi^\alpha := \xi_1^{\alpha_1} \ldots \xi_t^{\alpha_t},$$

where each $\xi_j^{\alpha_j}$ is itself a monomial

$$\xi_j^{\alpha_j} := \xi_{1j}^{\alpha_{1j}} \ldots \xi_{mj}^{\alpha_{mj}},$$

with $\xi_{ij}^0 = 1$ for any i, j. We interpret ξ^Λ as 1. A formal power series ψ is an infinite formal combination of the monomials ξ^α with coefficients in k. Since $\xi^\alpha = \xi^{\alpha 0} = \xi^{\alpha 00} = \ldots$, care must be taken not to count each ξ^α more than once. This is the reason for introducing Δ. Thus, a formal power series in the ξ_{ij} with coefficients in k is a map

$$\psi\colon \Delta \to k\colon \alpha \mapsto \psi_\alpha,$$

denoted also as

$$(5.1) \quad \psi(\xi_1, \xi_2, \dots) = \sum_{\alpha \text{ in } \Delta} \psi_\alpha \xi^\alpha, \quad \psi_\alpha \text{ in } k.$$

The set of formal power series can be made into a k-vector space by defining term-wise the addition of two power series ψ, $\hat{\psi}$ and the multiplication by scalars r in k:

$$(\psi + r\hat{\psi})_\alpha := \psi_\alpha + r\hat{\psi}_\alpha \quad \text{for all } \alpha.$$

Using the structure of the monoid $(\Delta, +)$, we may also induce a convolution product among series, which extends the multiplication of monomials defined as $\xi^\alpha . \xi^\beta := \xi^{\alpha+\beta}$. This extension is well-defined because $(\Delta, +)$ is locally finite, i.e. each α in Δ can be split in only finitely many ways as $\alpha_1 + \alpha_2$, α_i in Δ. The convolution of the power series ψ_1, ψ_2 is then defined globally by the formula:

$$(5.2) \quad (\psi\hat{\psi})_\alpha := \sum_{\beta+\gamma=\alpha} \psi_\beta \hat{\psi}_\gamma \quad \text{for all } \alpha \text{ in } \Delta.$$

The set of all formal power series forms a k-algebra when endowed with the operations of scalar product, sum, and convolution product. In fact, it is easy to prove that this algebra has no zero divisors.

We intend to derive response maps by evaluating power series for particular values of the ξ_{ij}. Thus we want to restrict our attention to a suitable class of series so that evaluation at arbitrary input values is welldefined. In accordance with related investigations in the literature, we shall call these Volterra series.

Let ψ be a formal power series. For each ξ_{ij}, ψ may be rearranged into a power series in ξ_{ij} whose coefficients are power series in the other variables. We call ψ a (formal) Volterra series (over k) iff, after each such rearrangement, ψ becomes a polynomial in ξ_{ij}.

In other words, ψ is a Volterra series precisely when there exist integers d_{ij} such that any ξ_{ij} appearing in ψ has exponent $\leq d_{ij}$. The smallest bound d_{ij} for the exponents of ξ_{ij} is the degree $\deg_{ij} \psi$ of ψ in ξ_{ij} (if ξ_{ij} does not appear in ψ, $\deg_{ij} \psi := -\infty$). Thus ψ is a Volterra series if and only if $\deg_{ij} \psi < \infty$ for all i, j.

For example,

$$\psi_1 := \xi_{11}^2 + \xi_{11}^2\xi_{12}^4 + \xi_{11}^2\xi_{12}^4\xi_{13}^8 + \ldots + \xi_{11}^2 \ldots \xi_{1n}^{2^n} + \ldots$$

is a Volterra series, and $\deg_{1j} \psi_1 = 2^j$ (if $m > 1$ then also $\deg_{ij} \psi_1 = -\infty$ for $i = 2, \ldots, m$);

$$\psi_2 := \xi_{11} + \xi_{12} + \xi_{12}^2 + \ldots + \xi_{1n} + \ldots + \xi_{1n}^n + \ldots$$

is also a Volterra series, with $\deg_{1j} \psi_2 = j$; but

$$\psi_3 := \xi_{11} + \xi_{11}^2 + \xi_{11}^3 + \ldots + \xi_{11}^n + \ldots$$

is not a Volterra series.

Since finiteness of the $\deg_{ij}$ is preserved under the algebra operations, the set of all Volterra series is a k-algebra. This k-algebra is an integral domain, since the algebra of all power series does not have any zero divisors.

(5.3) NOTATION. Ψ_k, or just Ψ, is the k-algebra of (formal) Volterra series over k.

(5.4) DEFINITION. The degree of a Volterra series ψ is

$$\deg \psi := \sup_{i,j} \{\deg_{ij} \psi\} \leq \infty.$$

Thus $\deg \psi_1 = \deg \psi_2 = \infty$ for the above examples while, on the other hand,

$$\deg (a_1\xi_{11} + a_2\xi_{12} + \ldots + a_n\xi_{1n} + \ldots) = 1.$$

A column p-vector of Volterra series can be obviously regarded as a power series in the ξ_{ij} with coefficients in k^p, via the identification

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \vdots \\ \psi^{(p)} \end{pmatrix} = \sum_{\alpha \text{ in } \Delta} \begin{pmatrix} \psi^{(1)}_\alpha \\ \vdots \\ \psi^{(p)}_\alpha \end{pmatrix} \xi^\alpha .$$

The definition of degree can be obviously generalized to the vector case via $\deg_{ij} \psi := \max \{\deg_{ij} \psi^{(1)}, \ldots, \deg_{ij} \psi^{(p)}\}$. We let Ψ^p denote the set of all vector Volterra series.

Volterra series with $\deg \psi < \infty$ will be studied in detail in Chapter V. An important tool in that study will be the concept of exponent series, which we now introduce.

The concept of time-shift is incorporated into the context of Volterra series through a product of Volterra series which is based upon the monoid $(\Delta, \cdot)$. We denote by $\psi.\hat{\psi}$ the Volterra series defined by

$$(5.5) \quad (\psi.\hat{\psi})_\alpha := \sum_{\beta\gamma=\alpha} \psi_\beta \hat{\psi}_\gamma \quad \text{for all } \alpha \text{ in } \Delta.$$

Note that $\psi.\hat{\psi}$ is not the same as the convolution product $\psi\hat{\psi}$ defined by (5.2). A change of notation is useful at this stage. Instead of writing $\psi = \Sigma\, \psi_\alpha \xi^\alpha$, we shall use the notation

$$(5.6) \quad \Sigma\, \psi_\alpha \alpha,$$

and call the expression (5.6) the exponent series φ associated to ψ. Thus φ is just a different notation for the same mathematical object ψ. If φ, $\hat{\varphi}$ are associated to ψ, $\hat{\psi}$, we denote $\varphi\hat{\varphi} := \psi.\hat{\psi}$. With these notations, the product (5.5) can now be expressed simply as a linear extension of the multiplication among indeterminates:

$$(5.7) \quad \varphi\hat{\varphi} = (\Sigma\, \varphi_\alpha \alpha)(\Sigma\, \hat{\varphi}_\beta \beta) = \sum_\alpha \sum_\beta \varphi_\alpha \hat{\varphi}_\beta \alpha\beta.$$

Exponent series provide a new way of expressing the condition "$\deg \psi < \infty$". For this, let

(5.8) $\operatorname{supp} \psi = \operatorname{supp} \varphi := \{\alpha \text{ in } \Delta \mid \psi_\alpha \neq 0\}$,

be the *support* of ψ (or of its associated exponent series).

(5.9) DEFINITION. *The support of* ψ *is finitely generated iff there exists a finite subset* $J = J_\psi$ *of* $\underline{\underline{N}}^m$ *such that*

(5.10) $\operatorname{supp} \psi \subseteq \Delta_J := J^* \cap \Delta$,

i.e., each column α_j *of* $\alpha = \alpha_1 \ldots \alpha_t$ *is in* J, *for each* α *in* $\operatorname{supp} \psi$.

In terms of the exponent series φ associated to ψ, (5.9) *means that* φ *is a power series in the finitely many (noncommuting) variables in* J. Since for any integer d there are only finitely many vectors in $\underline{\underline{N}}^m$ with all entries $\leq d$, we have the following trivial

(5.11) LEMMA. $\deg \psi < \infty$ *if and only if* $\operatorname{supp} \psi$ *is finitely generated*.

The fact that $\deg \psi < \infty$ is equivalent to φ being a series in finitely many variables will be exploited in Chapter V.

We now return to our study of arbitrary Volterra series. Their introduction was motivated by the need of evaluating series at arbitrary input values. We now study these evaluations. In fact, we study a more general type of operation on Volterra series.

Let K be an overring of k, and suppose given an infinite family $r = \{r_{ij},\ i = 1, \ldots, m,\ j = 1, 2, \ldots\}$ of elements of K. As before, we introduce the shorthand notation $r_j = r_{1j}, \ldots, r_{mj}$ for each j, and, for each $\alpha = \alpha_1 \ldots \alpha_t$ in Δ,

$$r^\alpha := r_1^{\alpha_1} \ldots r_t^{\alpha_t} := r_{11}^{\alpha_{11}} \ldots r_{mt}^{\alpha_{mt}},$$

(products in the ring K).

The definition of a Volterra series ψ as a power series with all $\deg_{ij} \psi < \infty$ is clearly equivalent to the requirement that ψ be a polynomial when expressed as a series in each finite subset of variables $\xi_1, \ldots, \xi_t$, for any fixed t. Thus we may write

$$(5.12)\quad \psi(\xi_1, \xi_2, \ldots) = \sum_{\substack{\alpha \text{ in } \Delta \\ |\alpha| \le t}} \zeta_\alpha(\xi_{t+1}, \ldots)\xi^\alpha,$$

and $\zeta_\alpha = 0$ for all but finitely many α. Let us make the substitutions $\xi_{ij} \mapsto r_{ij}$ in (5.12), for $i = 1, \ldots, m$ and $j = 1, \ldots, t$, performing the resulting products between the r_{ij}. We obtain

$$(5.13)\quad \psi(r_1, \ldots, r_t, \xi_{t+1}, \ldots) = \sum_{\substack{\alpha \text{ in } \Delta \\ |\alpha| \le t}} \zeta_\alpha(\xi_{t+1}, \ldots)r^\alpha.$$

Since, as we already remarked, the sum in (5.13) is finite, a further evaluation $\xi_{ij} \mapsto 0$, $i = 1, \ldots, m$, $j > t$, results in a finite linear combination

$$(5.14)\quad \psi(r_1, \ldots, r_t, 0, \ldots) = \sum \zeta_\alpha(0, \ldots)r^\alpha, \; \zeta_\alpha(0, \ldots) \text{ in } k.$$

When $K = k$, (5.14) is then in k, the result of substituting a finite "input" sequence into the Volterra series. In general we obtain an element of K, and the assignment

$$(5.15)\quad \Psi \to K: \psi \mapsto \psi(r_1, \ldots, r_t, 0, \ldots),$$

is clearly a k-algebra homomorphism.

We may instead apply to (5.13) the further substitutions $\xi_{ij} \mapsto \xi_{i,j-t}$, $i = 1, \ldots, m$, $j > t$, to obtain

$$(5.16)\quad \psi(r_1, \ldots, r_t, \xi_1, \xi_2, \ldots) = \sum \zeta_\alpha(\xi_1, \ldots)r^\alpha.$$

Since (5.16) is a finite K-combination of Volterra series over $k \subseteq K$, we may regard (5.16) as a Volterra series with coefficients in K; this justifies the notation $\psi(r_1, \ldots, r_t)(\xi_1, \xi_2, \ldots)$ or just

(5.17) $\psi(r_1, \ldots, r_t)$,

instead of $\psi(r_1, \ldots, r_t, \xi_1, \ldots)$. An alternative is to view $\psi(r_1, \ldots, r_t)$ as an element of $\Psi \otimes_k K$. The assignment

(5.18) $\Psi \to \Psi \otimes_k K\colon \psi \mapsto \psi(r_1, \ldots, r_t)$,

is clearly a k-algebra homomorphism. Moreover, (5.18) is an isomorphism when the r_{ij} are algebraically independent over Ψ, since it just amounts to a relabeling of variables.

In order to state a technical lemma to be used later, we need the following

(5.19) NOTATION. Let $t \geq 0$ be an integer. Then

$$\epsilon_t\colon \Psi \to k[\xi_1, \ldots, \xi_t],$$

is the homomorphism $\psi \mapsto \psi(\xi_1, \ldots, \xi_t, 0, \ldots)$.

Since $k[\xi_1, \ldots, \xi_t]$ is an integral domain, $\ker \epsilon_t$ is a prime ideal. Also,

(5.20) $\bigcap_{t\geq 0} \ker \epsilon_t = \{0\}$.

Let $s \leq t$. Then there exists an onto homomorphism

$$\grave{\epsilon}_{s,t}\colon k[\xi_1, \ldots, \xi_t] \to k[\xi_1, \ldots, \xi_s],$$

obtained by setting $\xi_{ij} = 0$ for $i = 1, \ldots, m$ and $j = s + 1, \ldots, t$. By definition of ϵ_t,

(5.21) $\epsilon_{s,t} \circ \epsilon_t = \epsilon_s$.

Let B be a k-subalgebra of Ψ and write $B_t := \epsilon_t(B)$. Then $R_t := \ker \epsilon_t|B = B \cap \ker \epsilon_t$ is a prime ideal of B and $\bigcap_t R_t \subseteq \bigcap_t \ker \epsilon_t = \{0\}$. We prove a technical

(5.22) LEMMA. $\operatorname{trdeg} B = \sup_{t\geq 0} \{\operatorname{trdeg} B_t\}$.

PROOF. By (4.2), trdeg $B \geq \sup\{\text{trdeg } B_t\}$. Thus it will be enough to prove that $\sup\{\text{trdeg } B_t\} = n < \infty$ implies $B \simeq B_t$ for all large t. Since each $\epsilon_{s,t}|B_t\colon B_t \to B_s$ is onto, the integers trdeg B_t form an ascending chain, bounded above by n. So trdeg B_r = trdeg $B_{r+1} = \dots$ for some r. By (4.2), $\epsilon_{r,t}|B_t$ is an isomorphism for each $t \geq r$. So by (5.21), $R_t = R_r$. Therefore $R_r = \bigcap R_t = \{0\}$ and $\epsilon_t|B\colon B \simeq B_t$ for all $t \geq r$. □

6. Construction of Ω and Γ.

We now begin to define response maps.

(6.1) DEFINITION. The input space Ω is the k-space $X(\Psi)$.

To verify that Ω is well-defined according to the setup developed in Chapter II, we remark that Ψ is k-reduced. Indeed, ker ϵ_t is a closed ideal for each t, since $k[\xi_1, \dots, \xi_t]$ is k-reduced. So by (5.20) the ideal $\{0\}$ is closed, i.e. Ψ is k-reduced.

(6.2) DEFINITION. The space of input values is

$$U := k^m.$$

The algebra of polynomial functions on the k-space U is $k[T_1, \dots, T_m]$. Therefore the algebra $A(U^t)$ of polynomial functions on the t-fold product of k-spaces $U^t = U \times \dots \times U$ is the ring of polynomials in mt variables. So we may denote $A(U^t)$ by $k[\xi_1, \dots, \xi_t]$. We adopt the notational convention of writing the sequences $(u_1, u_2, \dots, u_t)$ in U^t in an inverted order $(u_t, \dots, u_1)$. Thus, the coordinate function ξ_{ij} acts on elements of U^t by $\xi_{ij}(u_s, \dots, u_1) := u_{ij}$ = i-th entry of j-th vector counted from the right (for instance, if $m = 1$, then $\xi_{12}(0, 1, 0, 0, 0) = 0$, $\xi_{14}(0, 1, 0, 0, 0) = 1$). This notation will be consistent with the following interpretation: $(u_t, \dots, u_1)$ represents an input sequence such that u_j is the input at time $1 - j$.

Using this notation, ϵ_t gives rise to a closed embedding $i_t := X(\epsilon_t)\colon U^t \to \Omega$. Similarly, each $i_{s,t} := X(\epsilon_{s,t})\colon U^s \to U^t$ is a closed embedding, mapping sequences $(u_s, \dots, u_1)$ of U^s into $(0, \dots, 0, u_s, \dots, u_2, u_1)$ in U^t. By (5.21) the following diagram commutes:

(6.3)
$$\begin{tikzcd} U^s \arrow[rr, "i_{s,t}"] \arrow[dr, "i_s"'] & & U^t . \arrow[dl, "i_t"] \\ & \Omega & \end{tikzcd}$$

Therefore the sets $i_t(U^t)$, $t \geq 0$, form an ascending chain in Ω whose union (limit) may be identified with the set of all infinite sequences $(\dots, u_n, \dots, u_2, u_1)$ with finitely many nonzero entries; the rule

(6.4) $$(\dots, 0, u_t, \dots, u_1) \sim u_t z^{t-1} + \dots + u_1,$$

permits identifying this union with the set of polynomials over some symbol, say z, with coefficients in $U = k^m$. Thus we have the

(6.5) NOTATION. $U[z] := \bigcup_{t\geq 0} i_t(U^t)$.

We shall denote by (0) the sequence with $u_t = 0$ for all t.

(6.6) LEMMA. $\overline{U[z]} = \Omega$.

PROOF. Clear by (2.7.f) and (5.20). □

We may regard Ω *as a "completion" of* $U[z]$. *This will allow us to endow* $U[z]$ *with the geometric structure carried by a subset of the* k-*space* Ω.

There will be no danger in identifying U^t with $i_t(U^t)$, so that we may think of U^t as the closed subset of $U[z]$ corresponding to the sequences $(\dots, 0, u_t, \dots, u_1)$.

Now let K be the field obtained from k by adjunction of dennumerably many new indeterminates S_{ij}, $i = 1, \dots, m$, $j = 1, 2, \dots$. Applying (5.18) with $r_{ij} := S_{ij}$, we define

(6.7) $\theta_t : \Psi \to \Psi \otimes k[S_1, \dots, S_t] : \psi \mapsto \psi(S_1, \dots, S_t).$

By duality, there is a polynomial map

(6.8) $\delta_t := X(\theta_t) : \Omega \times U^t \to \Omega.$

Thus δ_t is the map whose transpose $A(\delta_t) = \theta_t$. We claim that the polynomial action of U^t upon Ω given by δ_t is the natural extension to Ω of the "concatenation" maps

(6.9) $$\delta_{s,t} : U^s \times U^t \to U^{s+t} : ((u_s, \dots, u_1), (\hat{u}_t, \dots, \hat{u}_1)) \mapsto$$
$$\mapsto (u_s, \dots, u_1, \hat{u}_t, \dots, \hat{u}_1).$$

In other words, we have:

(6.10) LEMMA. *The following diagram commutes for each* s, t:

$$\begin{tikzcd}
U^s \times U^t \arrow[r, "\delta_{s,t}"] \arrow[d, "i_s \times 1"] & U^{s+t} \arrow[d, "i_{s+t}"] \\
\Omega \times U^t \arrow[r, "\delta_t"] & \Omega .
\end{tikzcd}$$

PROOF. By duality, it is necessary and sufficient to verify that the following diagram commutes:

$$\begin{tikzcd}
\Psi \arrow[rr, "A(\delta_t)"] \arrow[d, "\epsilon_{s+t}"] & & \Psi[S_1, \dots, S_t] \arrow[d, "{\epsilon_s[S_1, \dots, S_t]}"] \\
k[\xi_1, \dots, \xi_{s+t}] \arrow[rr, "A(\delta_{s,t})"] & & k[\xi_1, \dots, \xi_s][S_1, \dots, S_t] .
\end{tikzcd}$$

Note that, in the coordinates displayed, $A(\delta_{s,t})$ is given by $\xi_{ij} \mapsto S_{ij}$ if $j = 1, \dots, t$, and by $\xi_{ij} \mapsto \xi_{i,j-t}$ if $j > t$. Thus the diagram commutes by definition of ϵ_s, ϵ_{s+t}, θ_t. □

In view of (6.10), there will be no danger in denoting the operations $\delta_{s,t}$, as well as δ_t, for all s and t, simply by concatenation

(6.11) $\omega v := \delta_t(\omega, v)$ for ω in Ω, v in U^t.

Let f be a polynomial function $\Omega \to k$. Since $A(\Omega) = \iota(\Psi) \simeq \Psi$, clearly f can be identified with a Volterra series $\psi = \psi_f$. So, by (3.8), $\psi|U^t = \psi \circ i_t$,

$$A(\psi \circ i_t)(T) = (A(i_t) \circ A(\psi))(T) = A(i_t)(\psi) = \epsilon_t(\psi);$$

thus (with the notation in (5.14))

(6.12) $\psi|U^t\colon (u_t, \ldots, u_1) \mapsto \psi(u_1, \ldots, u_t, 0, \ldots)$.

Since $U[z]$ is the increasing union of the U^t, a map $U[z] \to X$ is specified by its restrictions to the U^t. In particular, take ψ in Ψ. Then by (6.12) the value of $\psi|U[z]$ at $(\ldots, u_n, \ldots, u_1)$ is obtained by evaluating the power series ψ at $\xi_{ij} := u_{ij}$ = i-th row of u_j. This evaluation is well defined because almost all u_j are zero. A continuous function is already determined by its values in a dense subset, so by (6.6) the assignment $\psi \mapsto \psi|U[z]$ is one-to-one.

Thus the following mild abuse of terminology is justified:

(6.13) DEFINITION. A polynomial map $\ell\colon U[z] \to X$, where X is a k-space, is the restriction of a polynomial map $\ell^{\Omega}\colon \Omega \to X$.

The gist of the introduction of Ω is that the abstract set of input sequences $U[z]$ can now be exhibited as a dense subset of the k-space Ω and thus $U[z]$ is itself endowed (by restriction) with coordinates and polynomial functions.

Thus the polynomial functions $f\colon U[z] \to k$ are in a bijective correspondence with Volterra series ψ, via evaluation of ψ at $\xi_{ij} := u_{ij}$. More generally, a polynomial map $f\colon U[z] \to k^p$ is uniquely determined by the functions $\pi_j \circ f\colon U[z] \to k$, where π_j, $j = 1, \ldots, p$ are the natural projections $k^p \to k$. So polynomial maps $f\colon U[z] \to k^p$ are in a bijective correspondence with Ψ^p, the ordered p-tuples of Volterra series.

(6.14) DEFINITION. The space of output values is $Y := k^p$.

Thus,

(6.15) $\Psi^p \approx$ polynomial maps $U[z] \to Y$.

Finally, we define

$$U^* := \bigcup_{t \geq 0} U^t \quad \text{(disjoint union)}.$$

This set should not be confused with $U[z]$, the set of finitely nonzero sequences, which was obtained as a quotient set of U^*, via the identifications $(u_t, \ldots, u_1) \sim (0, \ldots, 0, u_t, \ldots, u_1)$. The unique element of U^0 is denoted $(\emptyset)$.

(6.16) DEFINITION. The output space Γ is the set Y^{U^*} of all maps $U^* \to Y$.

Thus an element of Γ is an U^*-indexed sequence of elements of Y. By (1.9) and (3.8), Γ is a k-space, the product of the k-space Y with itself, U^* times.

7. Abstract Response Maps and Systems.

For any vector space V over k we consider the set of sequences $\underline{\underline{Z}} \to V$ with support bounded on the left:

$$\underline{\underline{V}} := \{u: \underline{\underline{Z}} \to V \mid (\exists\, \tau_u)(u(t) = 0 \text{ if } t < \tau_u)\}.$$

The shift operator $\sigma = \sigma_V: \underline{\underline{V}} \to \underline{\underline{V}}$ is defined by

$$(\sigma u)(t) := u(t + 1) \text{ for all } t \text{ in } \underline{\underline{Z}}.$$

In particular we shall call

$\underline{\underline{U}}$:= set of input sequences;

$\underline{\underline{Y}}$:= set of output sequences.

(7.1) DEFINITION. *An input/output map* $\mathbf{f}$ *is a map* $\mathbf{f}: \mathbf{U} \to \mathbf{Y}$; $\mathbf{f}$ *is called*

(a) *(strictly) causal iff, for all* u, $\hat{u}$ *in* $\mathbf{U}$, *and for all* τ *in* $\mathbf{Z}$, $u(t) = \hat{u}(t)$ *for* $t < \tau$ *implies* $\mathbf{f}(u)(t) = \mathbf{f}(\hat{u})(t)$ *for* $t \leq \tau$;

(b) *constant structure (or shift-invariant) iff* $f(\sigma u) = \sigma f(u)$ *for all* u *in* $\mathbf{U}$.

An input/output pair of $\mathbf{f}$ *is a pair* $(\mathbf{f}(u), u)$, u *in* $\mathbf{U}$.

An input/output map can be regarded as an abstract description of the external behavior of a "black box" (physical device, computer, etc.) operating at discrete instants $\ldots, -2, -1, 0, 1, 2, \ldots$ of time. Constant structure means that this behavior is invariant under time shifts.

Rather than working with the input/output map, it is technically more convenient to work with the response map of the same "black box", i.e., with the description of outputs resulting immediately after the application of finite sequences of inputs.

(7.2) DEFINITION. *A response map* f *is a map*

$$f: U[z] \to Y.$$

Let $\eta: \mathbf{Y} \to Y: y \mapsto y(1)$; then *the assignments*

$$\mathbf{f} \mapsto f := \eta \circ \mathbf{f} | U[z],$$

and

$$f \mapsto \mathbf{f}(u)(t) := f(\sigma^t u),$$

establish a bijection between causal constant-structure input/output maps and response maps.

In the above formulas we are implicitely identifying $U[z]$ with a subset of $\mathbf{U}$, via the rule

$$u_\tau z^{\tau-1} + \ldots + u_1 \mapsto u,$$

where $u(-t) := u_{t-1}$ if $0 \leq t < \tau$ and $u(t) = 0$ otherwise.

In view of the above bijection, we shall state most of our definitions and results in terms of response maps. Only when dealing with input/output equations shall we refer again directly to input/output maps.

The following abstract definitions and notations are mostly well-known. They belong to "general system theory"; no structure is imposed on systems or preserved by response maps. Later we shall refine these definitions by adding suitable algebraic structure.

(7.3) DEFINITION. *An abstract (constant structure) system* Σ *is an object* $(X, P, h, x^\#)$, *where*

(a) X *is the state set;*

(b) $P: X \times U \to X$ *is the transition map;*

(c) $h: X \to Y$ *is the output map; and*

(d) $x^\#$ *in* X *is the initial state, and satisfies* $P(x^\#, 0) = 0$.

The t-th iterate of P *is the map* $P^{(t)}: X \times U^t \to X$ *defined recursively by*

$$P^{(0)} := 1_X, \quad P^{(1)} := P,$$
$$P^{(t+1)}(x, (u_{t+1}, \ldots, u_1)) := P(P^{(t)}(x, (u_{t+1}, \ldots, u_2)), u_1).$$

The reachability map of Σ *is*

$$g: U[z] \to X: u_t z^{t-1} + \ldots + u_1 \mapsto P^{(t)}(x^\#, (u_t, \ldots, u_1));$$

the t-*step reachability map is* $g_t := g|U^t$; *the* t-*step reachable set is* $X_t := g_t(U^t)$.

For each w *in* U^t, $t \geq 0$, *the observable map induced by* w *is*

$$h^w: X \to Y: x \mapsto h \circ P^{(t)}(\underline{x}, w).$$

The basic observables of Σ *are the functions* $h_j^w := \pi_j \circ h^w$, $j = 1, \ldots, p$, w in U^*. *The observability map of* Σ *is*

$$h^\Gamma: X \to \Gamma: x \mapsto \{h^w(x), \quad w \text{ in } U^*\}.$$

Σ *is called*

reachable iff g *is onto*;

observable iff h^Γ *is one-to-one*;

abstractly canonical iff both reachable and observable.

The response of Σ *is* $f_\Sigma := h \circ g$. *Given a response* f, Σ *realizes* f *iff* $f = f_\Sigma$.

(7.4) DEFINITION. *Consider abstract systems* Σ, $\hat{\Sigma}$. *An abstract system morphism* $T: \Sigma \to \hat{\Sigma}$ *is given by a map* $T: X \to \hat{X}$ *such that*:

(i) $T(x^\#) = \hat{x}^\#$, and

(ii) *the diagram*

$$\begin{tikzcd}
X \times U \arrow[r, "P"] \arrow[dd, "T \times 1"] & X \arrow[dd, "T"] \arrow[dr, "h"] & \\
 & & Y, \\
\hat{X} \times U \arrow[r, "\hat{P}"] & \hat{X} \arrow[ur, "\hat{h}"'] &
\end{tikzcd}$$

commutes.

Note that the existence of a system morphism $T: \Sigma \to \hat{\Sigma}$ implies that $f_\Sigma = f_{\hat{\Sigma}}$; this is proved by a trivial induction.

We shall call $\underline{\text{Syst}}_{\text{abs}}$ the category consisting of abstract systems as objects and morphisms defined as in (7.4).

Analogously with concepts related to Σ, the same concepts can be defined directly in terms of the response map:

(7.5) DEFINITION. *Let* f *be a response map. The observable map of* f *induced by* w, *where* w *is in* U^*, *is*

$$f^w\colon U[z] \to Y\colon v \mapsto f(vw).$$

The basic observables of f *are the functions* $f^w_j := \pi_j \circ f$, $j = 1, \ldots, p$, w *in* U^*. *The observability map of* f *is*

$$f^\Gamma\colon U[z] \to \Gamma\colon v \mapsto \{f^w(v),\ \ v \text{ in } U^*\}.$$

Thus the *value* of f^w is the result of an input/output experiment in which the output is observed at the end of the application of the concatenated input vw, v = given, w = chosen. The collection of results of *all* such experiments is f^Γ.

Since $f^{(0)} = f$, clearly

$$f = \hat{f} \ \text{ iff } \ f^\Gamma = \hat{f}^\Gamma.$$

In terms of the new notations, the definition of realization may be restated as follows. For any system Σ and any w in U^t, v in U^s,

$$\begin{aligned} f^v_\Sigma(w) = f_\Sigma(wv) = h\circ g(wv) &= h\circ P^{(s+t)}(x^\#, wv) = \\ &= h\circ P^{(s)}(g(w), v) = h^v\circ g(w), \end{aligned}$$

so $f^\Gamma_\Sigma = h^\Gamma\circ g$. Thus,

(7.6) Σ *realizes* f *iff* $f^\Gamma = h^\Gamma\circ g$.

We now restate in our formalism largely well known facts (see, for instance, EILENBERG [1974, Chapter XII]).

(7.7) LEMMA. *Let* $\Sigma = (X, P, h, x^\#)$ *and* $\hat{\Sigma} = (\hat{X}, \hat{P}, \hat{h}, \hat{x}^\#)$ *have the same response map. Assume that* Σ *is reachable and that* $\hat{\Sigma}$ *is observable. Then there exists a unique map* $T\colon X \to \hat{X}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
 & X \arrow[dd, "T"] \arrow[dr, "h^\Gamma"] & \\
U[z] \arrow[ur, "g"] \arrow[dr, "\hat{g}"'] & & \Gamma. \\
 & \hat{X} \arrow[ur, "\hat{h}^\Gamma"'] &
\end{tikzcd}$$

This unique T *induces an abstract system morphism* $\Sigma \to \hat{\Sigma}$.

PROOF. We first prove uniqueness. If such a T exists, then $\hat{h}^{\Gamma}(T(g(w))) = \hat{h}^{\Gamma}(\hat{g}(w))$ for all w in $U[z]$, so $T(g(w)) = \hat{g}(w)$ by observability of $\hat{\Sigma}$. Since Σ is reachable, every state in X is of the form $g(w)$. Thus there is a unique choice of T,

(*) $\quad T\colon g(w) \mapsto \hat{g}(w)$.

To prove existence, we need to see that T as in (*) is a well-defined map. In other words, if $g(w) = g(v)$, it should follow that $\hat{g}(w) = \hat{g}(v)$. But $\hat{h}^{\Gamma}(\hat{g}(w)) = h^{\Gamma}(g(w)) = h^{\Gamma}(g(v)) = \hat{h}^{\Gamma}(\hat{g}(v))$ implies $\hat{g}(w) = \hat{g}(v)$ by observability of $\hat{\Sigma}$.

We now prove that T is an abstract system morphism. Since $T(x^{\#}) = T(g(0)) = \hat{g}(0) = \hat{x}^{\#}$, we are only left to prove that $T(P(g(w), u)) = \hat{P}(T(g(w)), u)$ for every w in $U[z]$ and every u in U. But $T(P(g(w), u)) = T(g(wu)) = \hat{g}(wu) = \hat{P}(\hat{g}(w), u)$, as required. $\square$

(7.8) THEOREM. *Let* f *be a response map. Then* f *has an abstractly canonical realization* Σ_{ac}. *If* Σ *is any other abstractly canonical realization of* f *then there exists a unique isomorphism* $T\colon \Sigma_{ac} \to \Sigma$.

PROOF. Abstractly canonical realizations Σ correspond to factorizations $f^{\Gamma} = g_{\Sigma} \circ h_{\Sigma}^{\Gamma}$, where g_{Σ} is onto and h_{Σ}^{Γ} is one-to-one. We first prove the uniqueness part. Given any two abstractly canonical systems Σ and $\hat{\Sigma}$, there exist by (7.7) $T_1\colon \Sigma \to \hat{\Sigma}$, $T_2\colon \Sigma \to \hat{\Sigma}$. Since $T_1 \circ T_2$ is an abstract system morphism, it follows by the uniqueness part of (7.7) that $T_1 \circ T_2$ = identity abstract system morphism $\hat{\Sigma} \to \hat{\Sigma}$. Similarly, $T_2 \circ T_1$ is the identity $\Sigma \to \Sigma$. Thus T_1, T_2 are inverse abstract system isomorphisms.

For the existence part, simply take $X_{ac} := f^{\Gamma}(U[z]) \subseteq \Gamma$, $x_{ac}^{\#} := f^{\Gamma}(0)$, h_{ac} := projection onto $(\emptyset)$-th factor, and $P_{ac}(f^{\Gamma}(w), u) := f^{\Gamma}(wu)$. Note that P_{ac} is well defined, since $f^{\Gamma}(w) = f^{\Gamma}(\hat{w})$ implies that for any u, v, $f^{v}(wu) = f(wuv) = f^{uv}(w) =$
$= f^{uv}(\hat{w}) = f^{v}(\hat{w}u)$. $\square$

8. Polynomial Response Maps and k-Systems.

We begin now to study response maps which are polynomial (Definition (6.13)). The input/output map associated to a polynomial response map will be called a polynomial input/output map.

The notion of polynomial response maps is new, as is the following

(8.1) DEFINITION. The abstract constant structure system $\Sigma = (X, P, h, x^{\#})$ is a k-system iff X is a k-space and both P and h are polynomial maps.

The terminology "k-system" is just a convenient short name for our systems. In no way is it intended to imply that (8.1) constitutes the most general class of systems definable in terms of the field structure of k.

(8.2) LEMMA. The reachability map $g: U[z] \to X$ of a k-system Σ is a polynomial map.

PROOF. By definition (6.13), we need to find a polynomial extension $g^{\Omega}: \Omega \to X$ of g. Dually, we construct a homomorphism $A(g^{\Omega}): A(X) \to \Psi$ such that $\epsilon_t \circ A(g^{\Omega}) = A(g_t)$ for each $t \geq 0$.

Pick any a in $A(X)$. Then $A(g_t)(a) = a \circ g_t$ belongs to $k[\xi_1, \ldots, \xi_t]$ for each $t \geq 0$. We define a power series ψ^a in the variables ξ_{ij} by

$$\psi^a_{\alpha} := \text{coefficient of } \xi^{\alpha} \text{ in } A(g_{|\alpha|})(a),$$

for each α in Δ.

Now we claim that $\deg_{ij} \psi^a < \infty$ for each i, j, i.e., ψ^a is a Volterra series. Take any i, j. Let $t \geq j$. From the definition of $P^{(t)}$ it is clear that

$$P^{(t)} = P^{(j)} \circ (P^{(t-j)} \times 1_{U^j}).$$

As $g_t = P^{(t)}(x^{\#}, \cdot)$,

$$a \circ g_t = a \circ P^{(j)} \circ (P^{(t-j)}(x^{\#}, \cdot) \times 1_{U^j}).$$

Thus the degree to which ξ_{ij} appears in $a \circ g_t$, for any $t \geq j$, is at most the degree of the polynomial $a \circ P^{(j)}$ in the variable ξ_{ij}. So ψ^a is a Volterra series.

Let $A(g^{\Omega}) := a \mapsto \psi^a$. By construction, $\epsilon_t \psi^a = A(g_t)(a)$ for each a in $A(X)$ and $t \geq 0$. Since each $A(g_t)$ is a homomorphism, $A(g^{\Omega})$ is also a homomorphism. □

The following result, the main of this section, suggests that k-systems are the natural realizations of polynomial response maps. This will be confirmed later by the result on canonical realizations. We shall reserve the name "polynomial systems" for a special type of k-system in which a strong finiteness condition holds, which will allow P and h to be represented by actual polynomials.

(8.3) THEOREM. The response map f is polynomial if and only if f is realized by some k-system.

PROOF. ["only if"] The free realization $\Sigma_{\text{free}}(f) := (\Omega, \delta_1, f^{\Omega}, (0))$ is a k-system realizing f.

["if"]. Let $f = f_{\Sigma}$ for some k-system Σ. Define

$$f^{\Omega} := h \circ g^{\Omega}.$$

Then $f^{\Omega}|U[z] = h \circ g^{\Omega}|U[z] = h \circ g = f$. So f is polynomial. □

Properties of X serve to classify k-systems.

We shall say that Σ is a polynomial system [respectively almost polynomial] iff X_{Σ} is a variety [respectively an almost variety].

An irreducible Σ is one for which X_{Σ} is irreducible.

Similarly, we define (recall Section (4))

$$\dim \Sigma := \dim X_{\Sigma}.$$

The polynomial functions $X_\Sigma \to k$ are the *costates* of Σ.

The notion of abstract system morphism is too weak to serve for comparing k-systems. A suitable category **Syst**$_k$ of k-systems is obtained with morphisms as in the following

(8.4) DEFINITION. *An abstract system morphism* $T\colon \Sigma \to \hat{\Sigma}$ *between two k-systems is a k-system morphism iff* $T\colon X \to \hat{X}$ *is a polynomial map.* $T\colon \Sigma \to \hat{\Sigma}$ *is dominating, a closed embedding, etc., iff* $T\colon X \to \hat{X}$ has *the corresponding property;* Σ *dominates* $\hat{\Sigma}$ *iff there exists a dominating* $T\colon \Sigma \to \hat{\Sigma}$; Σ *is a closed subsystem of* $\hat{\Sigma}$ *iff there exists a closed embedding* $T\colon \Sigma \to \hat{\Sigma}$.

It is easy to see that k-systems form a category with the above notion of morphism. Note that a k-system isomorphism $T\colon \Sigma \simeq \hat{\Sigma}$ is the same as a polynomial change of coordinates in the state space.

9. Quasi-Reachability.

From here until the end of Chapter IV, f *is an arbitrary response map and* $\Sigma = (X, P, h, x^\#)$ *is an arbitrary* k-*system*.

The *reachable set* of Σ is

$$X_R := g(U[z]) = \bigcup_{t\geq 0} g(U^t) = \bigcup_{t\geq 0} X_t .$$

(9.1) DEFINITION. Σ *is quasi-reachable iff* $\overline{X}_R = X$.

By (6.6) and (3.11) we have the following

(9.2) LEMMA. *The following statements are equivalent:*

(a) Σ *is quasi-reachable*.

(b) g^Ω *is dominating*.

(c) $A(g^\Omega)$ *is one-to-one*.

(9.3) LEMMA (SONTAG and ROUCHALEAU [1975]). $\overline{X}_t = \overline{X}_{t+1}$ *for some* $t \geq 0$ *implies* $\overline{X}_t = \overline{X}_R$.

PROOF. Since P is polynomial, it is continuous; thus

$$X_{t+2} = P(X_{t+1} \times U) \subseteq P(\overline{X}_{t+1} \times U) = P(\overline{X}_t \times U) \subseteq \overline{P(X_t \times U)} = \overline{X}_{t+1},$$

Since clearly $X_{t+1} \subseteq X_{t+2}$, it follows that $\overline{X}_{t+1} \subseteq \overline{X}_{t+2} \subseteq \overline{\overline{X}}_{t+1} = \overline{X}_{t+1}$, so $\overline{X}_{t+1} = \overline{X}_{t+2}$ and the result follows by induction. □

(9.4) COROLLARY. *If* $\dim \Sigma = n < \infty$ *then* $\overline{X}_n = \overline{X}_R$.

PROOF. Since g^{Ω} is continuous and U^t is irreducible for each $t \geq 0$, $\overline{X}_t$ is irreducible. By (4.5), the chain $\{\overline{X}_t\}$ cannot have length greater than n. So (9.3) gives the desired result. □

Thus *in the finite-dimensional case a quasi-reachable system is quasi reachable in bounded time*. The analogous statement for reachability is false, as illustrated by the following example. Take $k := \underline{\underline{R}}$, $m = p := 1$, $X := \underline{\underline{R}}$, $x^{\#} := 0$, $P(x, u) := x + u^2 - 2u$, and h arbitrary. Then $X_t = \{x \text{ in } \underline{\underline{R}} \mid x \geq -t\} \neq X_R = X$ for all $t \geq 0$.

(9.5) LEMMA. Σ *has a quasi-reachable closed subsystem* Σ_Q.

PROOF. Let $X_Q := \overline{X}_R$. Since P is continuous, $P(X_Q \times U) \subseteq X_Q$. We may therefore define $\Sigma_Q := (X_Q, P|X_Q \times U, h|X_Q, x^{\#})$. The inclusion of $X_Q \subseteq X$ exhibits Σ_Q as a closed subsystem of Σ. □

10. Algebraic Observability.

As discussed in intuitive terms in KALMAN [1968, Chapter 10], observability of Σ means the existence of a procedure for determining the state x of Σ from data obtained by experiments of the type: "apply an input sequence to Σ beginning in state x and observe the corresponding output sequence". In terms of the basic observables $\{h_j^w\}$ introduced in (7.3), this informal description of observability can be made precise by requiring the existence of a set of arbitrary functions of experiments

$$\eta_\lambda(h_{j_1}^{w_1,\lambda}, \ldots, h_{j_{r(\lambda)}}^{w_{r(\lambda)},\lambda}),$$

with $\Lambda = \{\lambda\}$ some arbitrary indexing set, such that each state x is uniquely determined by the data $\{\eta_\lambda(x)\}_{\lambda\in\Lambda}$.

When this procedure is interpreted in the weakest possible, nonconstructive sense, the functions η_λ are completely arbitrary and "observability" reduces to the abstract definition (7.3). In the case of (finite-dimensional) linear systems over a field this abstract definition turns out to be equivalent to the existence of linear combinations η_λ which give every coordinate of the state; see KALMAN [1968, Chapter 10]. For linear systems over a commutative ring, however, the abstract notion of observability is no longer equivalent to the existence of a linear procedure; some of the resulting problems are studied in SONTAG [1976, 1977]. In general, observability should be formalized with reference to the particular category over which the system in question is defined. Thus, in the context of k-systems observability is defined by requiring that each coordinate of the state (i.e., every costate of the system) be a polynomial in the basic observables. This is the definition given below, which is a direct generalization of that given in SONTAG and ROUCHALEAU [1975] for polynomial systems. A direct study of bilinear response maps, KALMAN [1976] suggests the same conclusion.

(10.1) DEFINITION. The observation space $\underset{\sim}{L}(\Sigma)$ [respectively the observation algebra $\underset{\sim}{A}(\Sigma)$] is the linear subspace [respectively the subalgebra] of $A(X)$ generated by the basic observables. An observable is a costate in $\underset{\sim}{A}(\Sigma)$. Σ is algebraically observable iff $\underset{\sim}{A}(\Sigma) = A(\Sigma)$. When X is irreducible, the observation field $Q(\Sigma)$ is the quotient field of $\underset{\sim}{A}(\Sigma)$.

Consider the observability map $h^\Gamma : X \to \Gamma = Y^{U^*}$ introduced in (7.3). Since each h^w is a polynomial map, h^Γ is also a polynomial map. By (1.9), $A(\Gamma)$ is generated by the algebras $A(Y)$ appearing in the coproduct. So the image $A(h^\Gamma)(A(\Gamma))$ coincides with the algebra generated by the $A(h^w)(A(Y))$, each of which is itself generated by $h_1^w, \ldots, h_p^w$. We conclude that $\underset{\sim}{A}(\Sigma)$ is the image of $A(\Gamma)$. The dual of this fact is:

(10.2) LEMMA. Σ *is algebraically observable iff* h^{Γ} *is a closed embedding*. □

(10.3) COROLLARY. *If* Σ *is algebraically observable then* Σ *is (abstractly) observable*.

Algebraic observability is a stronger requirement than abstract observability. This is clear from the counterexamples given in (3.12).

We remarked in (9.5) that every k-system has a closed quasi-reachable subsystem. It is less trivial to prove the corresponding statement for algebraic observability:

(10.4) PROPOSITION. Σ *dominates an algebraically observable system* Σ^{obs}.

PROOF. Let $i\colon \underset{\sim}{A}(\Sigma) \to A(X)$ be the inclusion map. Let $X^{obs} := X(\underset{\sim}{A}(\Sigma))$, $x^{\#obs} := A(i)(x^{\#})$. Since $A(h)(A(Y)) \subseteq \underset{\sim}{A}(X)$, we may factor $h\colon X \to Y$ as

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "X(i)"'] & & Y, \\
& X^{obs}, \arrow[ur, "h^{obs},"'] &
\end{tikzcd}$$

for some h^{obs}.

Thus the proof will be complete if we can prove that P induces through $A(i)$ a k-system morphism $P^{obs}\colon X^{obs} \times U \to X^{obs}$; then $X(i)$ is the required dominating k-system morphism $\Sigma \to \Sigma^{obs}$. Therefore we must show that

(10.5) $A(P)(\underset{\sim}{A}(\Sigma)) \subseteq \underset{\sim}{A}(\Sigma)[T_1, \ldots, T_m]$.

i.e., we must prove that when an element q of $A(P)(\underset{\sim}{A}(\Sigma))$ is expressed as a polynomial in the variables $T_1, \ldots, T_m$, the coefficients of such a polynomial q are again in $\underset{\sim}{A}(\Sigma)$. Since the algebra $\underset{\sim}{A}(\Sigma)$ is generated by the space $\underset{\sim}{L}(\Sigma)$, statement (10.5) follows from the following

(10.6) LEMMA. *Let* $\hat{\mathbf{L}}_t$ *be the subspace of* $A(X)$ *generated by all those* h_j^w *with* w *in* U^t. *Then*

$$A(P)(\hat{\mathbf{L}}_t) \subseteq \hat{\mathbf{L}}_{t+1}[T_1, \dots, T_m].$$

PROOF. We first observe that when $T_1, \dots, T_m$ are specialized at an u in $U = k^m$, a polynomial in $A(P)\hat{\mathbf{L}}_t$ becomes an element of $\hat{\mathbf{L}}_{t+1}$. Indeed, denote by $e(u)\colon A(X)[T_1, \dots, T_m] \to k$ the corresponding specialization. Then

$$e(u) \circ A(P) \circ A(h^w) = A(h^{uw}).$$

Therefore $e(u)(A(P)\hat{\mathbf{L}}_t) \subseteq \hat{\mathbf{L}}_{t+1}$, as wanted. Our claim follows from the following more general result (with $A = A(X)$, $F = k[T_1, \dots, T_m]$ and the $\{c_i\}$ a finite set of monomials in $T_1, \dots, T_m$):

(10.7) MAIN LEMMA, PART 1. *Let* A, F *be vector spaces over* k, *with* F *a space of functions* $Z \to k$ *for some set* Z. *Assume that* $c_1, \dots, c_n$ *are linearly independent elements of* F, *and let* $a_1, \dots, a_n$ *be in* A. *Then the linear subspace of* A *generated by*

$$\{\sum_{i=1}^{n} c_i(z)a_i, \quad z \text{ in } Z\},$$

coincides with the subspace generated by $a_1, \dots, a_n$.

PROOF. Clearly $\sum c_i(z)a_i$ is in the space generated by $a_1, \dots, a_n$. It is then enough to prove that each a_i, say a_1, can be written as

$$a_1 = \sum_{j=1}^{n} \lambda_j (\sum_{i=1}^{n} c_i(z_j)a_i),$$

for some $z_1, \dots, z_n \in X$. Rewriting this expression as

$$a_1 = \sum_{i=1}^{n} a_i (\sum_{j=1}^{n} \lambda_j c_i(z_j)),$$

we see that it is enough to prove the existence of a λ in k^n such that $T\lambda = (1, 0, 0, \ldots, 0)'$, where

$$T = \begin{pmatrix} c_1(z_1) & c_1(z_2) & \cdots & c_1(z_n) \\ \vdots & \vdots & & \vdots \\ c_n(z_1) & c_n(z_2) & \cdots & c_n(z_n) \end{pmatrix}.$$

It is therefore enough to find $z_1, \ldots, z_n$ such that T is nonsingular.

Form the $n \times Z$ matrix $\hat{T}$ whose i-th row is c_i seen as an element of k^Z. Then existence of T (as a submatrix of $\hat{T}$) follows from the fact that rank $T = n$. □

11. *Existence and Uniqueness of Canonical Realizations*.

(11.1) DEFINITION. Σ *is canonical iff* Σ *is quasi-reachable and algebraically observable*.

We associate yet another map to f. The *extended observability map* $f^{\Omega\Gamma}: \Omega \to \Gamma$ of f is the observability map of the system $\Sigma_{\text{free}}(f)$ introduced in (8.3). The observability map $f^{\Gamma}: U[z] \to \Gamma$ introduced in (7.5) is clearly the restriction of $f^{\Omega\Gamma}$ to $U[z]$. Since $U[z]$ is dense in Ω, we may immediately generalize (7.6):

(11.2) Σ *realizes* f *iff* $f^{\Omega\Gamma} = h^{\Gamma} \circ g^{\Omega}$.

(11.3) LEMMA. *Let* $\Sigma = (X, P, h, x^{\#})$ *be a quasi-reachable and* $\hat{\Sigma} = (\hat{X}, \hat{P}, \hat{h}, \hat{x}^{\#})$ *an algebraically observable* k-*system which realize* f. *Then there exists a unique* k-*system morphism* $T: \Sigma \to \hat{\Sigma}$.

PROOF. Consider the diagram

(11.4)
$$\begin{tikzcd} & X \arrow[dd, dashed, "T"] \arrow[dr, "h^{\Gamma}"] & \\ \Omega \arrow[ur, "g^{\Omega}"] \arrow[dr, "\hat{g}^{\Omega}"'] & & \Gamma. \\ & \hat{X} \arrow[ur, "\hat{h}^{\Gamma}"'] & \end{tikzcd}$$

By hypothesis, g^{Ω} is dominating, $\hat{h}^{\Gamma}$ is a closed embedding, and $h^{\Gamma} \circ g^{\Omega} = f^{\Omega\Gamma} = \hat{h}^{\Gamma} \circ \hat{g}^{\Omega}$. Thus by the dual of (1.5) there exists a polynomial map $T\colon X \to \hat{X}$ making (11.4) commutative. Restricting to $U[z] \subseteq \Omega$, the diagram

$$\begin{tikzcd}
 & X_R \arrow[dr, "(h|X_r)^{\Gamma}"] \arrow[dd, "T|X_R"] & \\
U[z] \arrow[ur, "g"] \arrow[dr, "\hat{g}"'] & & \Gamma, \\
 & \hat{X} \arrow[ur, "\hat{h}^{\Gamma}"'] &
\end{tikzcd}$$

commutes. Thus we may apply (7.7) to the abstract systems $(X_R, P|X_r \times U, h|X_R, x^{\#})$ and $\hat{\Sigma}$. We conclude that the continuous maps $T \circ P\colon X \times U \to \hat{X}$ and $\hat{P} \circ (T \times 1_u)$ coincide in the dense subset $X_R \times U$, so they are equal. □

The main result of this chapter is:

(11.5) THEOREM. *Let* f *be a polynomial response map. Then there is a canonical* k-*system* Σ_f *realizing* f. *If* $\hat{\Sigma}$ *is any other canonical* k-*system realizing* f, *there is a unique* k-*system isomorphism* $T\colon \Sigma_f \to \hat{\Sigma}$.

PROOF. Uniqueness is clear by (11.4). To prove existence, take the system $\Sigma_{\text{free}}(f)$; this is quasi-reachable because g^{Ω} is the identity. Applying (10.4) we obtain the observable system $\Sigma_f := (\Sigma_{\text{free}}(f))^{\text{obs}}$. Since $\Sigma_{\text{free}}(f)$ dominates Σ_f, the latter is also quasi-reachable. □

CHAPTER IV. FINITENESS CONDITIONS

We have shown in the previous chapter that any polynomial response map f is realizable by a canonical k-system. We now turn to studying what conditions must f satisfy in order that the canonical system Σ_f has various finiteness properties.

The main tool in this study will be three structures obtained from the basic observables (7.5) of f by different algebraic operations: the observation space $\underset{\sim}{L}_f$, algebra $\underset{\sim}{A}_f$ and field $\underset{\sim}{Q}_f$. We show that the conditions

$\underset{\sim}{Q}_f$ = finitely generated field over k,

$\underset{\sim}{A}_f$ = finitely generated algebra over k,

$\underset{\sim}{L}_f$ = finitely generated vector space over k,

each corresponds to an important characterization of Σ_f.

We then relate each of the above conditions to the existence of certain input/output equations for f.

We also show how to check the finiteness condition on $\underset{\sim}{Q}_f$ via a Jacobian criterion. As an application we show that

$$y(t) = u(t - 1) + u(t - 2)^2 + \dots + u(t - \ell)^\ell + \dots ,$$

has no possible finite-dimensional realization.

In the final section we discuss examples and counterexamples associated to the results and constructions of the last two chapters.

We continue to denote by f an arbitrary polynomial response map.

12. *The Observables of* f.

Since Σ_f is quasi-reachable and Ω is irreducible, it follows that Σ_f is irreducible, so $Q(A(X_f))$ is well-defined:

(12.1) DEFINITION. *The observation space* $\underset{\sim}{L}_f$ *[respectively observation algebra* $\underset{\sim}{A}_f$, *respectively observation field* $\underset{\sim}{Q}_f$*] of* f *is*

$\mathbf{L}(\Sigma_f)$ [*respectively* $\mathbf{A}(\Sigma_f)$, *respectively* $\mathbf{Q}(\Sigma_f)$].

Thus $\mathbf{L}_f$, $\mathbf{A}_f$, $\mathbf{Q}_f$ are the space, algebra, and field generated by the basic observables $\{(f^{\Omega})^{w}_{j},\ w \text{ in } U^{*},\ j = 1, \ldots, p\}$ of $\Sigma_{\text{free}}(f)$. Each function $(f^{\Omega})^{w}_{j}\colon \Omega \to k\colon \omega \mapsto \pi_j(f(\omega w))$ is already determined by its restriction f^{w}_{j} to $U[z]$, which is dense in Ω. The restriction $(f^{\Omega})^{w}_{j} \mapsto f^{w}_{j}$ serves to establish the following identifications in terms of the basic observables of f introduced in (7.5):

(12.2) $\mathbf{L}_f$ *is the subspace of* $k^{U[z]}$ *generated by the basic observables* f^{w}_{j}, w *in* U^{*}, $j = 1, \ldots, p$.

(12.3) $\mathbf{A}_f$ *is the subalgebra of* $k^{U[z]}$ *generated by* $\mathbf{L}_f$.

The f^{w}_{j} can be also viewed as maps $U^{*} \to k$, so one can also identify $\mathbf{L}_f$ and $\mathbf{A}_f$ with the subspace and subalgebra of $k^{U^{*}}$ generated by the f^{w}_{j}.

Yet another representation of the observables is obtained via the (vector) Volterra series $\psi_f = (\psi_f^{(1)}, \ldots, \psi_f^{(p)})'$ of f^{Ω} (see (6.15)). By the discussion in (5.16)-(5.17), the Volterra series of $f^{u_t \ldots u_1}$ is precisely $\psi_f(u_1, \ldots, u_t)$; coordinatewise:

(12.4) *The Volterra series of* $f_j^{u_t \ldots u_1}$ *is* $\psi_f^{(j)}(u_1, \ldots, u_t)$.

Thus $\mathbf{L}_f$, $\mathbf{A}_f$, and $\mathbf{Q}_f$ can be interpreted as the space, algebra, and field generated by the series $\{\psi_f^{(j)}(u_1, \ldots, u_t),\ j = 1, \ldots, p,\ u_t \ldots u_1 \text{ in } U^{*}\}$. Using this interpretation, we may define

(12.5) $\deg_{ij} f := \deg_{ij} \psi_f, \quad \deg f := \deg \psi_f.$

By (12.4) it follows that

(12.6) $\deg_{ij} f^{w} \leq \deg_{i,j+t} f, \quad \deg f^{w} \leq \deg f,$

for w in U^{t}.

The observables are the main system invariants in our approach. The study of $\mathbf{L}_f$, $\mathbf{A}_f$, $\mathbf{Q}_f$ will be simplified by the consideration of various chains which approximate them:

(12.7) DEFINITION. *The observability chains* $\{\mathbf{L}^{0,t}(\Sigma),\ t \geq 0\}$ *and* $\{\mathbf{A}^{0,t}(\Sigma),\ t \geq 0\}$ *of* Σ *are, respectively, the subspaces and subalgebras* of $\mathbf{A}(\Sigma)$ *generated for each* t *by the elementary observables* h_j^w, $j = 1, \dots, p$, w in U^r, $r \leq t$. *The reachability chains* $\{\mathbf{L}^{R,t}(\Sigma),\ t \geq 0\}$ and $\{\mathbf{A}^{R,t}(\Sigma),\ t \geq 0\}$ *are, respectively, the subspaces and subalgebras generated for each* t *by the restrictions* $h_j^w|X_t$, $j = 1, \dots, p$, w *in* U^*. *The diagonal chains* $\{\mathbf{L}^t(\Sigma),\ t \geq 0\}$ *and* $\{\mathbf{A}^t(\Sigma),\ t \geq 0\}$ *are, respectively, the subspaces and subalgebras generated for each* t *by the restrictions* $h_j^w|X_t$, $j = 1, \dots, p$, w *in* U^r, $r < t$. *When* Σ *is irreducible there are corresponding chains of fields* $Q^{0,t}(\Sigma)$, *etc. The observability, reachability, and diagonal chains of* f *are* $L^{0,t} := \mathbf{L}^{0,t}(\Sigma_f)$, *etc*.

We write $\mathbf{L}^{0,t}$, etc., instead of $\mathbf{L}^{0,t}(\Sigma)$, etc., when there is no danger of confusion. In terms of Volterra series, we have:

(12.8) $\mathbf{L}_f^{0,t}$, $\mathbf{A}_f^{0,t} \subseteq \Psi$ *are generated by all* $\psi_f(u_1, \dots, u_r)$, $r \leq t$;

(12.9) $\mathbf{L}_f^{R,t}$, $\mathbf{A}_f^{R,t} \subseteq k[\xi_1, \dots, \xi_t]$ *are generated by all* $\epsilon_t\psi_f$;

(12.10) $\mathbf{L}_f^t$, $\mathbf{A}_f^t \subseteq k[\xi_1, \dots, \xi_t]$ *are generated by all* $\epsilon_t(\psi_f(u_1, \dots, u_r))$, $r < t$.

Before proving some properties of the chains just introduced, we need the

(12.11) MAIN LEMMA, PART 2. *Let* A *be a* k-*algebra, and take a polynomial* Q *in* $A[T_1, \dots, T_r]$, *for some* $r \geq 0$. *Let* D *be a dense subset of* k^r. *Then the linear subspace of* A *spanned by* $\{Q(u),\ u$ *in* $D\}$ *is equal to the linear span of* $\{Q(u),\ u$ *in* $k^r\}$.

PROOF. We may assume that $Q \neq 0$. Write

$$Q(T_1, \dots, T_r) = \sum_{i=1}^{s} c_i(T_1, \dots, T_r)a_i,$$

with all c_i in $k[T_1, \dots, T_r]$ and all a_i in A, and with s smallest possible. Minimality of s implies that both $\{a_1, \ldots, a_s\}$ and $\{c_1, \ldots, c_s\}$ are linearly independent over k.

We claim that the restrictions $c_i|D$ are linearly independent as functions from D into k. Otherwise there would exist $b_1, \ldots, b_s$ in k such that $\sum b_i c_i(d) = 0$ for all d in D. By continuity of the polynomial function $\sum b_i c_i \colon k^r \to k$, it follows that $\sum b_i c_i = 0$ on all of k^r, contradicting linear independence of the c_i.

Main Lemma (10.7) can then be applied twice, first with $Z := D$ and then with $Z := k^r$. Thus $\{Q(u),\ u \text{ in } D\}$ and $\{Q(u),\ u \text{ in } k^r\}$ both span the same space as $\{a_1, \ldots, a_s\}$. □

We collect below some rather technical facts which will be needed in deriving the main results of this chapter.

(12.12) PROPOSITION. *For any* Σ *and* f,

(a) *If* $\mathbf{L}^{0,t}(\Sigma) = \mathbf{L}^{0,t+1}(\Sigma)$ *for some* $t > 0$, *then* $\mathbf{L}^{0,t}(\Sigma) = \mathbf{L}(\Sigma)$.

(b) *If* $\mathbf{A}^{0,t}(\Sigma) = \mathbf{A}^{0,t+1}(\Sigma)$ *for some* $t \geq 0$, *then* $\mathbf{A}^{0,t}(\Sigma) = \mathbf{A}(\Sigma)$.

(c) *If* $\mathbf{A}^{0,t+1}(\Sigma)$ *is algebraic over* $\mathbf{A}^{0,t}(\Sigma)$ *for some* $t \geq 0$ *and if* $\{P(x, u),\ x$ *in* $X,\ u$ *in* $U\}$ *is dense in* X (*e.g., if* Σ *is quasi-reachable), then* $\mathbf{A}(\Sigma)$ *is algebraic over* $\mathbf{A}^{0,t}(\Sigma)$.

(d) *If* $\mathbf{A}^{0,t+1}(\Sigma)$ *is integral over* $\mathbf{A}^{0,t}(\Sigma)$ *for some* $t \geq 0$, *then* $\mathbf{A}(\Sigma)$ *is integral over* $\mathbf{A}^{0,t}(\Sigma)$.

(e) $\mathbf{L}_f^{0,t}$ *is finite dimensional and* $\mathbf{A}_f^{0,t}$ *is a finitely generated algebra for all* $t \geq 0$.

(f) *If* $\dim \Sigma = n$ *then* $\dim \Sigma_{f_\Sigma} \leq n$.

(g) *Let* $\dim \Sigma_f = n$. *Then* $\epsilon_n|\mathbf{L}_f\colon\mathbf{L}_f \simeq \mathbf{L}_f^{R,n}$, and $\epsilon_n|\mathbf{A}_f\colon\mathbf{A}_f \simeq \mathbf{A}_f^{R,n}$.

(h) $\dim \Sigma_f = \sup_{t \geq 0} \{\operatorname{trdeg} \mathbf{A}_f^t\}$.

PROOF. (a) By (10.6),

$$(12.13)\quad \begin{aligned} A(P)(\underset{\sim}{L}^{0,r}) &= A(P)(\bigcup_{0\le s\le r} \hat{\underset{\sim}{L}}_s) = \bigcup_{0\le s\le r} A(P)(\hat{\underset{\sim}{L}}_s) \subseteq \\ &\subseteq \bigcup_{0\le s\le r} \hat{\underset{\sim}{L}}_{s+1}[T_1, \ldots, T_m] \subseteq \\ &\subseteq \underset{\sim}{L}^{0,r+1}[T_1, \ldots, T_m], \end{aligned}$$

for all $r \geq 0$. By the argument in (10.6), $\hat{\underset{\sim}{L}}_{t+2}$ is spanned by the coefficients of the polynomials in

$$A(P)(\hat{\underset{\sim}{L}}_{t+1}) \subseteq A(P)(\underset{\sim}{L}^{0,t+1}) = A(P)(\underset{\sim}{L}^{0,t}) \subseteq \underset{\sim}{L}^{0,t+1}[T_1, \ldots, T_m],$$

(using (10.3)). Thus $\hat{\underset{\sim}{L}}_{t+2}$ is spanned by elements of $\underset{\sim}{L}^{0,t+1}$, i.e $\hat{\underset{\sim}{L}}_{t+2} \subseteq \underset{\sim}{L}^{0,t+1}$. Then

$$\underset{\sim}{L}^{0,t+2} = \underset{\sim}{L}^{0,t+1},$$

and (a) follows by induction.

(b) Analogous to (a).

(c) Take any elementary observable h_j^w with w in U^{t+1}. Since, by hypothesis, h_j^w is algebraic over $\underset{\sim}{A}^{0,t}$, there exists an $s \geq 0$ and a polynomial $L(T_o, \ldots, T_s)$ in $k[T_o, \ldots, T_s]$ with the properties:

(i) $L(h_j^w, h_{j_1}^{v_1}, \ldots, h_{j_s}^{v_s}) = 0$

for some elementary observables $h_{j_1}^{v_1}, \ldots, h_{j_s}^{v_s}$, v_i in U^{r_i}, $r_i \leq t$, and (ii) if $\hat{L}(T_1, \ldots, T_s)$ is the leading coefficient of L expressed as a polynomial in T_o, then

$$(12.14)\quad \hat{L}(h_{j_1}^{v_1}, \ldots, h_{j_s}^{v_s}) \neq 0.$$

If

$$\hat{L}(h_{j_1}^{uv_1}(x), \ldots, h_{j_s}^{uv_s}(x)) = \hat{L}(h_{j_1}^{v_1}(P(x, u)), \ldots, h_{j_s}^{v_s}(P(x, u)) = 0$$

for all u in U and x in X, the density hypothesis leads to a contradiction of (12.14). So

$$D := \{u \text{ in } U \mid \hat{L}(h_{j_1}^{uv_1}, \ldots, h_{j_s}^{uv_s}) \neq 0\} \neq \emptyset,$$

and by (2.4), D is open, hence dense. Take u in D. Then

$$L(h_j^{uw}, h_{j_1}^{uv_1}, \ldots, h_{j_s}^{uv_s}) = L(h_j^{w}(P(\cdot, u)), \ldots, h_{j_s}^{v_s}(P(\cdot, u))) = 0,$$

and the equation is nontrivial, i.e.

$$\hat{L}(h_{j_1}^{uv_1}, \ldots, h_{j_s}^{uv_s}) \neq 0,$$

by definition of D. Thus the elementary observables

$$\{h_j^{uw}, u \text{ in } D, j = 1, \ldots, p\},$$

are algebraic over $\underset{\sim}{A}^{0,t+1}$. Let $Q(T_1, \ldots, T_m)$ be the polynomial $h_j^w \circ P$ considered as a polynomial in $T_1, \ldots, T_m$ (input variables) with coefficients in $A(X)$. We showed above that the elements $Q(u)$, u in D, are algebraic over $\underset{\sim}{A}^{0,t+1}$. By (12.11), the span of $\{Q(u), u \text{ in } k^m\}$ coincides with the span of $\{Q(u), u \text{ in } D\}$. Thus every generator $Q(u) = h_j^{uw}$ of $\underset{\sim}{A}^{0,t+2}$ is algebraic over $\underset{\sim}{A}^{0,t+1}$ for all uw in U^{t+2}. The result follows by induction on t.

(d) This is analogous to (c).

(e) Since $\underset{\sim}{A}_f^{0,t}$ is generated by $\underset{\sim}{L}_f^{0,t}$ = span of $\hat{\underset{\sim}{L}}_f^{0,0}, \ldots, \hat{\underset{\sim}{L}}_f^{0,t}$, it is enough to prove that each $\hat{\underset{\sim}{L}}_f^{0,0}$ is finite dimensional. But, by the Main Lemma (10.7), $\hat{\underset{\sim}{L}}_f^{0,t}$ is generated by the coefficients of $\psi_f^{(j)}(s_1, \ldots, s_t)$.

(f) By (9.5) and (10.4), Σ has a subsystem Σ_Q which dominates Σ_f (i.e. Σ_f is a "subquotient" of Σ). Thus (e) follows from (4.2).

(g) Apply (9.3) to Σ_f; observe that $A(g_n) = \epsilon_n$.

(h) By (c) above applied to Σ_f,

$$\dim \Sigma_f = \sup_t \{\operatorname{trdeg} \underset{\sim}{A}_f^{0,t}\}.$$

The result is then clear by (5.22) applied to each $\underset{\sim}{A}_f^{0,t}$. □

13. Finite Realizability and Minimality.

(13.1) DEFINITION. f *is finitely realizable iff* f *has a finite dimensional realization*.

We collect various characterizations of finite realizability in the following

(13.2) THEOREM. *The following statements are equivalent:*

(a) f *is finitely realizable*.

(b) $\dim \Sigma_f < \infty$.

(c) Σ_f *is an almost-polynomial system*.

(d) $\underset{\sim}{Q}_f$ *is a finitely generated field extension of* k.

(e) $\epsilon_t\colon \underset{\sim}{A}_f \simeq \underset{\sim}{A}_f^{R,t}$ *for some* $t \geq 0$.

(f) $\underset{\sim}{Q}_f = \underset{\sim}{Q}_f^{0,t}$ *for some* $t \geq 0$.

(g) $\underset{\sim}{A}_f$ *is algebraic over* $\underset{\sim}{A}_f^{0,t}$ *for some* $t \geq 0$.

PROOF. We prove (a) ⇒ (b) ⇒ (e) ⇒ (c) ⇒ (d) ⇒ (g) ⇒ (f) ⇒ (b) ⇒ ⇒ (a).

(a) ⇒ (b) This is (12.12e).

(b) ⇒ (e) This is (12.12f).

(e) ⇒ (c) By definition, $\underset{\sim}{A}^{R,t}$ is a subalgebra of $A(U^t) = k[\xi_1, \ldots, \xi_t]$.

(c) ⇒ (d) Let $\underset{\sim}{A}_f$ be included in the finitely generated algebra $k[b_1, \ldots, b_s]$. Then $\underset{\sim}{Q}_f$ is included in the finitely generated field $k(b_1, \ldots, b_s)$. By LANG [1965, Chapter X, Exercise 6], $\underset{\sim}{Q}_f$ is itself finitely generated.

(d) ⇒ (g) Clear from $\mathbf{Q}_f = \bigcup_{t \geq 0} \mathbf{Q}_f^{0,t}$.

(g) ⇒ (f) If $\mathbf{A}_f$ has the same quotient field as $\mathbf{A}_f^{0,t}$, then $\mathbf{A}_f$ is obviously algebraic over $\mathbf{A}_f^{0,t}$.

(f) ⇒ (b) Clear from (12.12e)

(b) ⇒ (a) Trivial. $\square$

We consider briefly the topic of minimal-dimensional realizations:

(13.3) DEFINITION. *A system* Σ *is minimal iff*

$$\dim \Sigma \leq \dim \hat{\Sigma}$$

for any $\hat{\Sigma}$ *with* $f_{\hat{\Sigma}} = f_{\Sigma}$.

By (12.12f), Σ *is minimal iff* $\dim \Sigma = \dim \Sigma_{f_\Sigma}$.

For any x in X we define the *observability class* of x,

(13.4) $\operatorname{Obs}(x) := \{z \text{ in } X \mid h^\Gamma(z) = h^\Gamma(x)\}$.

We are interested in the case in which $\operatorname{Obs}(x)$ is generically finite:

(13.5) DEFINITION. Σ *is weakly observable iff there exists an open dense subset* U *of* X *and an integer* $s \geq 0$ *such that* $\operatorname{Obs}(x) \cap U$ *has cardinality* $\leq s$ *for all* x *in* U. Σ *is weakly canonical iff* Σ *is quasi-reachable and weakly observable*.

Let Σ be a quasi-reachable realization of f. By (10.4), there is a dominating morphism $T\colon \Sigma \to \Sigma_f = (X_f, P_f, x_f^\#, h_f)$. Since T is a system morphism, $h_f^\Gamma \circ T = h^\Gamma$. Thus

$$h^\Gamma(x) = h^\Gamma(z) \quad \text{iff} \quad h_f^\Gamma(T(x)) = h_f^\Gamma(T(z)),$$

iff (observability of Σ_f!) $T(x) = T(z)$. So

$$\operatorname{Obs}(x) = T^{-1}(T(x)) = \text{fiber of } T \text{ through } x.$$

If either $k = \underline{\underline{R}}$ or k is algebraically closed, and if Σ is a quasi-reachable almost-polynomial system, then (4.6) implies that

(13.6) $\dim \text{Obs}(x) = \dim \Sigma - \dim \Sigma_f$

for all x in some open dense set U. In particular,

(13.7) THEOREM. Let either $k = \underline{\underline{R}}$ or let k be algebraically closed. The almost polynomial system Σ is weakly canonical if and only if Σ is both minimal and irreducible.

PROOF. ["if"] If Σ is not quasi-reachable, then Σ_Q has strictly lower dimension, contradicting minimality. So Σ is quasi-reachable, and there exists $T: \Sigma \to \Sigma_{f_\Sigma}$ dominating. Apply (4.6), noting that $n = m$. Let $U := j_X(X_1)$; then (4.6c) shows that the sets $\text{Obs}(x) \cap U$ have a bounded cardinality when x is in U.

["only if"] Irreducibility is a consequence of quasi-reachability. Assume that $\dim \Sigma_{f_\Sigma} < \dim \Sigma$. By (13.6),

$$\dim(\text{Obs}(x) \cap U) = \dim \overline{(\text{Obs}(x) \cap U)} = \dim \text{Obs}(x) > 0$$

for almost all x, so $\text{Obs}(x) \cap U$ cannot be generically finite. □

14. Polynomial Canonical Systems.

We have proved in the previous section that "f finitely realizable" is equivalent to "Σ_f is almost polynomial". We now ask under what (stronger) conditions Σ_f is a polynomial system.

The precise "input/output" condition for Σ_f = polynomial is immediate at this stage. Since $A(\Sigma_f) = \underset{\sim}{A}_f$, we see that the canonical system Σ_f is polynomial iff the observation algebra $\underset{\sim}{A}_f$ is a finitely generated k-algebra. Some useful characterizations of this latter condition are given in the next

(14.1) THEOREM. The following statements are equivalent:

(a) Σ_f *is a polynomial system*.

(b) $\underset{\sim}{A}_f^{0,t} = \underset{\sim}{A}_f$ *for some* $t \geq 0$.

(c) $\underset{\sim}{A}_f$ *is integral over* $A_f^{0,t}$ *for some* $t \geq 0$.

PROOF. (a) $\Rightarrow$ (b) Since

$$\underset{\sim}{A}_f = \bigcup_{t\geq 0} \underset{\sim}{A}_f^{0,t},$$

there is a $t \geq 0$ such that $\underset{\sim}{A}_f^{0,t}$ contains all the generators of $\underset{\sim}{A}_f$.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) $\underset{\sim}{A}_f$ integral over $\underset{\sim}{A}_f^{0,t}$ implies $\underset{\sim}{Q}_f = Q(\underset{\sim}{A}_f)$ is algebraic over $Q(\underset{\sim}{A}_f^{0,t})$, so trdeg $\underset{\sim}{Q}_f$ = trdeg $\underset{\sim}{Q}_f^{0,t}$ = finite by (12.12e). By (b) $\Leftrightarrow$ (d) in (13.2), Q_f is finitely generated over k. By (1.13a) we conclude that $\underset{\sim}{A}_f$ is finitely generated. □

We shall now describe a procedure which enables one to obtain a polynomial canonical realization when a finite set of generators of $\underset{\sim}{A}_f$ is known.

Assume now that $\underset{\sim}{A}_f$ is generated by $\psi_1, \ldots, \psi_n$ as a subalgebra of Ψ. In particular trdeg $\underset{\sim}{A}_f \leq n$, so $\epsilon_n|\underset{\sim}{A}_f$ is an isomorphism. Thus any relation

$$(14.2) \qquad \psi_i(u) = \sum_\alpha P_{\alpha,i}(\psi_1, \ldots, \psi_n)u^\alpha,$$

where $u = (u_1, \ldots, u_m)'$ is in U and the $P_{\alpha,i}$ are finitely many polynomials, is equivalent to

$$\epsilon_n\psi_i(u) = \sum_\alpha P_{\alpha,i}(\epsilon_n\psi_1, \ldots, \epsilon_n\psi_n)u^\alpha.$$

The transition map of the canonical realization Σ_f is given by $X(\theta_1)$, with θ_1 defined as in (6.7), θ_1: $\underset{\sim}{A}_f \to \underset{\sim}{A}_f[S_1]$: $\psi \mapsto \psi(S_1)$. Thus the canonical realization may be obtained from $\epsilon_{n+1}\psi_1, \ldots, \epsilon_{n+1}\psi_n$ as follows:

STEP 1. Substitute $\xi_{i1} \mapsto T_i$ and $\xi_{ij} \mapsto \xi_{i,j-1}$, $i = 1, \ldots, m$, $j \geq 2$, in each $\epsilon_{n+1}\psi$.

STEP 2. Write $\epsilon_{n+1}\psi_i$ as a polynomial in the variables $T = T_1, \ldots, T_m$:

(14.3) $$\epsilon_{n+1}\psi_i = \sum_{\alpha} q_{i\alpha}(\xi_1, \ldots, \xi_n)T^{\alpha}.$$

STEP 3. Express each $q_{i\alpha}$ as a polynomial combination of $\epsilon_n\psi_1, \ldots, \epsilon_n\psi_n$:

(14.4) $$q_{i\alpha} = P_{i\alpha}(\epsilon_n\psi_1, \ldots, \epsilon_n\psi_n);$$

this is always possible by (10.5), since $\epsilon_n\psi_1, \ldots, \epsilon_n\psi_n$ generate $\underset{\sim}{A}_f^{R,n}$.

STEP 4. Since each $\psi_f^{(j)}$ is in $\underset{\sim}{A}_f^{R,n}$, $j = 1, \ldots, p$, there are polynomials h_j with

$$\epsilon_n\psi_f^{(j)} = h_j(\epsilon_n\psi_1, \ldots, \epsilon_n\psi_n), \quad j = 1, \ldots, p.$$

Then Σ_f is the quasi-reachable subsystem Σ_Q of $\Sigma = (k^n, (P_{i\alpha}), (h_1, \ldots, h_p), 0)$, i.e., Σ_f *is given by the equations*

(14.5) $$\begin{cases} x_i(t+1) = \sum P_{i\alpha}(x_1(t+1), \ldots, x_n(t+1)), & x_i^{\#} := 0, \\ & i = 1, \ldots, n, \\ y_j(t) = h_j(x_1(t), \ldots, x_n(t)), \quad j = 1, \ldots, p, & \end{cases}$$

subject to the constraints

$$Q(x_1(t), \ldots, x_n(t)) = 0,$$

for each polynomial Q *for which* $Q(\epsilon_n\psi_1, \ldots, \epsilon_n\psi_n) = 0$.

(14.6) EXAMPLE. A simple illustration of the above procedure is the following. Let $f = f_{\Sigma}$, where Σ is given by $m = p = 1$, $X := k$ and

$$x(t+1) = x^2(t) + u(t), \quad x^{\#} := 0, \quad y(t) = x^2(t).$$

Thus, $f(u_t, \ldots, u_1) = ((\ldots((u_t^2 + u_{t-1})^2 + u_{t-2})^2 + \ldots + u_2)^2 + u_1)^2$. Note that $f^u = (f + u)^2$, or equivalently, $\psi_f(u) = (\psi_f + u)^2$ for all u in U. Thus ψ_f generates $\underset{\sim}{A}_f$, and $n = 1$. Since $\epsilon_2\psi_f = (\xi_2^2 + \xi_1)^2$, step 1 gives $(\xi_1^2 + T)^2$. But $\epsilon_1\psi_f = \xi_1^2$, so $(\xi_1^2 + T)^2 = (\epsilon_1\psi_f + T)^2$. Thus (12.5) becomes

$$(14.7) \qquad x(t + 1) = (x(t) + u)^2, \quad x^\# = 0, \quad y(t) = x(t),$$

with $X_f = k$. The procedure $\Sigma \to \Sigma_f$ has identified the pairs of unobservable states $\{x, -x\}$, giving the algebraically observable system Σ_f. On the other hand, (complete) reachability of Σ became just quasi-reachability of Σ_f. A more direct way of obtaining Σ_f in those cases (as with the present example) in which a quasi-reachable realization Σ is already known is through (10.4). By (12.12b) and (14.1c) we must generate the algebras $\underset{\sim}{A}^{0,t}(\Sigma)$ until $\underset{\sim}{A}^{0,t}(\Sigma) = \underset{\sim}{A}^{0,t+1}(\Sigma)$ for some t. Calculating, $\underset{\sim}{A}^{0,0}(\Sigma)$ = algebra generated by $h = k[\eta^2]$; $\underset{\sim}{A}^{0,1}(\Sigma)$ = algebra generated by η^2 and by all $(\eta^2 + u)^2$, u in k = (by 10.7) algebra generated by η^2 and by the coefficients in u of $\eta^4 + 2\eta^2 u + u^2 = k[\eta^2, \eta^4] = k[\eta^2] = \underset{\sim}{A}^{0,0}(\Sigma)$. So $\underset{\sim}{A}(\Sigma) = \underset{\sim}{A}^{0,0}(\Sigma)$. Restricting $A(P)$ to $\underset{\sim}{A}(\Sigma)$ one obtains again (14.7). □

We shall see in Section (1.6) that Σ_f is polynomial for a wide class of response maps, which includes all those maps defined by recursive polynomial equations.

15. Bounded Maps.

An important rôle in studying the condition $\dim \underset{\sim}{L}_f < \infty$ is played by the response maps introduced in the following

(15.1) DEFINITION. f *is bounded iff* $\deg f < \infty$.

(Recall Definitions (5.4) and (12.5) for deg f.)

The system-theoretic meaning of a bounded map is that no input is raised to a power higher than a certain bound. Very simple systems can give rise to *non*bounded maps. For instance, let

$$\Sigma \ : \ x(t+1) = x^2(t) + u(t); \quad x^{\#} := 0, \quad y(t) = x(t).$$

Computing f_Σ by iteration shows that it is clearly not bounded. However, the notion of bounded map is general enough to accommodate those classes of response maps for which a realization theory exists in the literature today. These cases correspond to the next three examples:

(15.2) EXAMPLE. f is *linear* iff $E_o\psi_f = 0$ and each monomial in ψ_f has degree = 1. In particular, $\deg f = 1 < \infty$. $\square$

(15.3) EXAMPLE (KALMAN [1969, 1976]). f is *multilinear* of order m iff all the nonzero monomials appearing in ψ_f are of the form

$$a\xi_{1i_1}^{\alpha_{1i_1}} \cdots \xi_{mi_m}^{\alpha_{mi_m}}, \quad a \text{ in } k,$$

i.e. $f|U^t$ is m-linear as a map $U^t = (k^m)^t = (k^t)^m \to Y$, for all $t \geq 0$. Then, $\deg f = 1 < \infty$. (More generally one may consider s-linear maps $f\colon U_1^t \times \ldots \times U_s^t \to Y$, where $U_i := k^{r_i}$, $i = 1, \ldots, s$ and $r_1 + \ldots + r_s = m$. Here we took all $r_i = 1$ just for notational simplicity.) $\square$

(15.4) EXAMPLE (BROCKETT [1972], ISIDORI and RUBERTI [1973], ISIDORI [1973, 1974], FLIESS [1973], D'ALESSANDRO, ISIDORI and RUBERTI [1974]). f is *internally-bilinear* iff all monomials appearing in ψ_f are of the type

$$a\xi_{i_1 1}^{\alpha_{i_1 1}} \cdots \xi_{i_t t}^{\alpha_{i_t t}}, \quad t \geq 0, \quad a \text{ in } k.$$

Again $\deg f = 1 < \infty$. $\square$

Note the difference between the bilinearity in (15.4) with that in (15.3) (with $m = 2$). Neither of these implies the other.

(15.5) THEOREM. *Let* f *be a response map. The following statements are equivalent:*

(a) f *is bounded and finitely realizable*.

(b) f *is bounded and realizable by a polynomial system*.

(c) f *is bounded and* Σ_f *is a polynomial system (i.e.,* $\mathbf{A}_f$ *is finitely generated)*.

(d) f *is bounded and* $\mathbf{L}_f$ *is finite dimensional*.

(e) $\dim \mathbf{L}_f < \infty$.

(f) f *admits a realization with* $X = k^n$ *and transitions of the form*

$$x(t+1) = F(u(t))x(t) + G(u(t)),$$

where $F(\cdot)$ *is a matrix, and* $G(\cdot)$ *a vector, of polynomial functions*.

(g) f *admits a realization as above with, further,* h *linear*.

(h) f *admits an observable realization as in* (g).

(i) f *admits a realization as in (h), with* $G = 0$.

PROOF. The only nontrivial implications in the diagram

$$\begin{tikzcd}
 & (d) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & (e) \arrow[r, Rightarrow] & (i) \arrow[dr, Rightarrow] & \\
(a) \arrow[ur, Rightarrow] & & (c) \arrow[dl, Rightarrow] & & (h), \arrow[dl, Rightarrow] \\
 & (b) \arrow[ul, Rightarrow] & (f) \arrow[l, Rightarrow] & (g) \arrow[l, Rightarrow] &
\end{tikzcd}$$

are (a) ⇒ (d), (e) ⇒ (i), and (f) ⇒ (a), which we now prove.

(a) ⇒ (d) Let $\dim \Sigma_f = n$ and $\deg f = d$. By (12.12g), $\epsilon_t\colon \mathbf{L}_f \simeq \mathbf{L}_f^{R,n}$. By (12.6), $\deg f^w \le d$ for all w in U^*. So $\deg \epsilon_t \psi_{f,j}(w) \le d$ for all w in U^* and $j = 1, \ldots, p$. Thus $\mathbf{L}_f^{R,n}$ is a linear space generated by a set of polynomials in nm variables of joint degree $\le nmd$. Thus $\dim \mathbf{L}_f = \dim \mathbf{L}_f^{R,n} \le (nm)^{nmd} < \infty$.

(e) ⇒ (i) Let $\dim \mathbf{L}_f = n$, and let $\{\psi_1, \ldots, \psi_n\}$ be a basis of $\mathbf{L}_f$ as a subspace of Ψ. By (11.5) the polynomials $q_{i\alpha}$ introduced in (14.3) are in $\mathbf{L}_f^{0,R}$. Thus each $P_{i\alpha}$ in (14.4) can be chosen *linear*, and (i) follows from the formulas (14.5).

(f) $\Rightarrow$ (a) Let $F(u) = (F_{ij}(u))$, a matrix of polynomials, and let $h(u) = (h_1, \ldots, h_p)$, a vector of polynomials. Denote $\deg F := \sup_{i,j} \{\deg F_{ij}\}$ and $\deg h := \sup_j \{\deg h_j\}$. Then

$$\deg f \leq \deg F . \deg h < \infty . \qquad \square$$

We can see from either (f) or (g) above that the *essential characteristic of bounded maps is the absence of any intrinsic nonlinear feedback*.

The above theorem shows that the problem of deciding whether a bounded map is realizable is equivalent to checking whether a realization as in (g), (h), or (i) exists. We shall study in Chapter V the connection between such special realizations and some questions of automata theory.

These special configurations are highly appealing because they can be studied by linear-algebraic methods. It should be noted carefully, however, that, for certain questions like synthesis, alternative (lower dimensional) representations may be more useful. This is illustrated by the following example. Consider the one-dimensional system Σ:

$$(15.6) \quad x(t+1) = x(t) + u(t), \quad x^{\#} := 0, \quad y(t) = x^s(t),$$

where $s \geq 1$ is an arbitrary integer. Then $\dim \underline{L}_{f_\Sigma} = s$ and therefore (see Chapter V) representations as in (g), (h), or (i) have dimension at least s. The system Σ may be, however, obtained back from any observable realization $\hat{\Sigma}$ as in (h) in the following way. Since Σ is canonical (easy verification) by (11.3) Σ is a closed subsystem of $\hat{\Sigma}$. Indeed, the reachable set of $\hat{\Sigma}$ will be one-dimensional, and the restrictions of $\hat{P}$ and $\hat{h}$ to this reachable set will define a subsystem isomorphic to Σ.

16. Input/Output Equations.

We shall now study the existence of input/output equations and relate them to various finiteness conditions. We treat only the case $p = 1$, since when there is more than one output channel one may separately study each $\pi_j \circ f$, $j = 1, \ldots, p$. We let S_{ij} be indeterminates, $i = 1, \ldots, m$, $j \geq 1$, and denote $S_j := S_{1j}, \ldots, S_{mj}$.

(16.1) DEFINITION. *Let* $r \geq 0$ *be an integer, and let* E *be a polynomial in* $k[L_0, L_1, \ldots, L_r, S_r, \ldots, S_1]$, *nontrivial in* L_0. *A pair* (y, u), y *in* $\mathbf{Y}$, u *in* $\mathbf{U}$ *satisfies* E *iff*

$$E(y(t), y(t-1), \ldots, y(t-r), u(t-1), \ldots, u(t-r)) = 0$$

for all t *in* $\mathbf{Z}$. *The response map* f *satisfies the algebraic difference equation* E *iff every input/output pair of* f *satisfies* E. *The order of the equation* E *is* r. *The equation* E *is*

(a) *rational when* $\deg_{L_0} E = 1$, *i.e.*

$$E = E_1(L_1, \ldots, L_r, S_r, \ldots, S_1)L_0 + E_2(L_1, \ldots, L_r, S_r, \ldots, S_1);$$

(b) *integral when*

$$E = E_1(S_r, \ldots, S_1)L_0^s + E_2(L_0, L_1, \ldots, L_r, S_r, \ldots, S_1),$$

with $\deg_{L_0} E_2 < s$;

(c) *recursive when rational and integral, i.e.*

$$E = E_1(S_r, \ldots, S_1)L_0 + E_2(L_1, \ldots, L_r, S_r, \ldots, S_1);$$

(d) *affine when*

$$E = \sum_{\ell=0}^{r} E_\ell(S_r, \ldots, S_1)L_j + \hat{E}(S_r, \ldots, S_1).$$

The main result of this section is the following

(16.2) THEOREM. *Let* f *be a polynomial response map.*

(a) *The following statements are equivalent:*

(i) f *is finitely realizable;*

(ii) f *satisfies an algebraic difference equation*;

(iii) f *satisfies a rational difference equation*.

(b) *The following statements are equivalent*:

(i) f *is bounded and finitely realizable*;

(ii) f *satisfies an affine difference equation*.

(c) *If* f *satisfies an integral difference equation then* Σ_f *is a polynomial system and* f *satisfies a recursive equation*.

The proof of this theorem will be postponed until we establish some technical facts about algebraic difference equations.

The study of such equations will be algebraized through the introduction of the field $K = k(\{S_{ij},\ i = 1, \ldots, m,\ j \geq 1\})$ obtained by adjoining the indeterminates S_{ij} to k. Let $\mathbf{L}^K_f$, $\mathbf{A}^K_f$ be the K-subspace and K-subalgebra of Ψ_K generated by ψ_f and (recall (5.17)) all the $\psi_f(S_t, \ldots, S_1)$ (note order of arguments!), for all $t \geq 0$. Let $\mathbf{Q}^K_f := \mathbf{Q}(A^K_f)$.

(16.3) LEMMA. *Let* f *be a polynomial response map. Then*

(a) f *satisfies an algebraic difference equation if and only if* $\operatorname{trdeg}_K Q^K_f < \infty$.

(b) f *satisfies a rational difference equation if and only if* $\mathbf{Q}^K_f$ *is a finitely generated field extension of* K.

(c) f *satisfies a recursive difference equation if and only if* $\mathbf{A}^K_f$ *is a finitely generated* K-*algebra*.

(d) *If* f *satisfies an integral equation and a (possibly different) rational equation then* f *satisfies a recursive equation*.

(e) f *satisfies an affine difference equation if and only if* $\dim_K L^K_f < \infty$.

PROOF. We begin with some general remarks. Observe that an equation E of order r is equivalent to

$$(16.4)\quad E(f^{u_r\cdots u_1}, f^{u_r\cdots u_2}, \ldots, f^{u_r}, f, u_1, \ldots, u_r) = 0;$$

for all $(u_r, \ldots, u_1)$ in U^r. In terms of Volterra series, (16.4) is equivalent to

$$(16.5)\quad E(\psi_f(S_r, \ldots, S_1), \psi_f(S_{r-1}, \ldots, S_1), \ldots, \psi_f(S_1), \psi_f, S_r, \ldots, S_1) = 0.$$

A change of variables $S_\ell \mapsto S_{\ell+1}$, $\ell = 1, \ldots, r$, $\xi_1 \mapsto S_1$, and $\xi_j \mapsto \xi_{j-1}$ if $j > 1$ transforms (16.5) into

$$(16.6)\quad E(\psi_f(S_{r+1}, \ldots, S_1), \psi_f(S_r, \ldots, S_1), \ldots, \psi_f(S_2, S_1), \psi_f(S_1), S_{r+1}, \ldots, S_2) = 0.$$

Thus an equation of order r gives rise to an equation of order $r + 1$.

We now prove (a). Let f satisfy E, with E of order r smallest possible. Let $c(L_1, \ldots, L_r, S_r, \ldots, S_1) \neq 0$ be the leading coefficient of E as a polynomial in L_o. Suppose that

$$(*)\qquad c(\psi_f(S_{r-1}, \ldots, S_1), \ldots, \psi_f, S_r, \ldots, S_1) = 0.$$

Dividing by S_r if necessary, we may assume that $c(L_1, \ldots, L_r, 0, S_{r-1}, \ldots, S_1) \neq 0$. Thus

$$c(\psi_f(S_{r-1}, \ldots, S_1), \ldots, \psi_f, 0, S_{r-1}, \ldots, S_1) = 0;$$

is an equation for f of order $\leq r - 1$, contradicting minimality of r. Therefore (*) is false. So (14.4) is a nontrivial equation for $\psi_f(S_r, \ldots, S_1)$ with coefficients in

$$\hat{K}_r := K(\psi_f, \ldots, \psi_f(S_{r-1}, \ldots, S_1)).$$

Since the derivation of (16.5) from (16.4) is just a relabeling of variables, the same argument gives a dependency for $\psi_f(S_{r+1}, \ldots, S_1)$ over $K(\psi_f, \ldots, \psi_f(S_r, \ldots, S_1))$. Since algebraic extensions of algebraic extensions are again algebraic (formula (4.1)) we conclude that

$\psi_f(S_{r+1}, \ldots, S_1)$ is also algebraic over $\hat{K}_r$, and by induction also $\underset{\sim}{Q}_f^K$ is algebraic over $\hat{K}_r$. Thus

$$\operatorname{trdeg}_K \underset{\sim}{Q}_f^K = \operatorname{trdeg}_K \hat{K}_r \leq r < \infty .$$

Conversely, assume that $\operatorname{trdeg}_K \underset{\sim}{Q}_f^K < \infty$. Since the $\psi_f(S_t, \ldots, S_1)$ generate $\underset{\sim}{Q}_f^K$, there exists an $r \geq 1$ such that $\underset{\sim}{Q}_f^K$ is algebraic over $\hat{K}_r$. In particular, $\psi_f(S_r, \ldots, S_1)$ is algebraic over $\hat{K}_r$, so there is an equation of algebraic dependence $Q(\psi_f(S_r, \ldots, S_1)) = 0$ with coefficients in $\hat{K}_r$. Multiplying by a common denominator if necessary, we may assume that all coefficients of Q are in $k[\{S_{ij}\} \cup \{\psi_f, \ldots, \psi_f(S_{r-1}, \ldots, S_1)\}]$. Dividing if necessary, we may further assume that no S_{ij} divides Q. Therefore the evaluation $S_{ij} \mapsto 0$ for all $j > r$ gives an algebraic equation of order r.

(b) A rational equation E is equivalent to the statement "$\psi_f(S_r, \ldots, S_1)$ belongs to $\hat{K}_r$". Since $\underset{\sim}{Q}_f^K$ is the union of the $\hat{K}_r$, (b) is clear.

(c) Existence of a recursive equation of order r is equivalent to $\psi_f(S_r, \ldots, S_1)$ belonging to the algebra

$$A_r = K[\psi_f, \ldots, \psi_f(S_{r-1}, \ldots, S_1)],$$

which is in turn equivalent to $\underset{\sim}{A}_f^K$ being finitely generated.

(d) An integral equation corresponds to $\psi_f(S_r, \ldots, S_1)$ being integral over A_r, so by induction as in passing from (16.5) to (16.6), we conclude that $\underset{\sim}{A}_f^K$ is integral over A_r. Since, by (b), $\underset{\sim}{Q}_f^K = Q(\underset{\sim}{A}_f^K)$ is finitely generated over K, we conclude from (1.13a) that $\underset{\sim}{A}_f^K$ is finitely generated.

(e) Similar to (c); just observe that an affine equation of order r is equivalent to $\psi_f(S_r, \ldots, S_1)$ belonging to the span of $\psi_f, \ldots, \psi_f(S_{r-1}, \ldots, S_1)$ as a space over K. □

(16.7) LEMMA. Let f be a polynomial response map. Then

(a) $\operatorname{trdeg}_K \mathbf{Q}_f^K < \infty$ *if and only if* $\operatorname{trdeg} \mathbf{Q}_f < \infty$.

(b) $\dim_K \mathbf{L}_f^K < \infty$ *if and only if* $\dim \mathbf{L}_f < \infty$.

(c) *If* $\mathbf{A}_f^K$ *is a finitely generated* K-*algebra, then* $\mathbf{A}_f$ *is finitely generated (over* k).

PROOF. We first prove that the finiteness statements on $\mathbf{L}_f^K$, $\mathbf{A}_f^K$, and $\mathbf{Q}_f^K$ imply the corresponding conditions on $\mathbf{L}_f$, $\mathbf{A}_f$, and $\mathbf{Q}_f$. By (6.3) these statements are equivalent to the existence, respectively, of an affine, recursive or algebraic equation E for f. Let $c(L_1, \ldots, L_r, S_r, \ldots, S_1)$ be the leading coefficient of E as a polynomial in L_0. Let

$$D := \{(u_r, \ldots, u_1) \text{ in } U^r \mid c(\psi_f(u_{r-1}, \ldots, u_1), \ldots, \psi_f(u_1), \ldots, \psi_f, u_r, \ldots, u_1) = 0\}.$$

By (6.4), $\psi_f(u_r, \ldots, u_1)$ is in the space $\mathbf{L}_f$ [respectively in the algebra $\mathbf{A}_f$, respectively algebraic over $\mathbf{Q}_f$] for all $(u_r, \ldots, u_1)$ not in D. The conclusion is then clear from the Main Lemma, Part II (12.11) and (12.12e).

Now we prove the "if" parts in (a) and (b). By the Main Lemma (10.7), the coefficients of $\psi_f(S_r, \ldots, S_1)$, considered as a polynomial in $S_1, \ldots, S_r$, are in $\mathbf{L}_f$. Thus $\mathbf{L}_f^K$ is in the linear space over K generated by $\mathbf{L}_f$, which has a finite basis by hypothesis. Since $\mathbf{Q}_f^K$ is generated as a field by the elements of $\mathbf{L}_f^K$, the conclusion about $\mathbf{Q}_f^K$ is also immediate. □

We may now complete the

PROOF OF (16.2). (a) Follows from (16.3a,b), (16.7a) and the equivalence of (a) and (d) in (13.2).

(b) Clear from (16.3e) and (16.7b).

(c) An integral equation is in particular an algebraic equation, so by (a) f satisfies a rational equation. The conclusion is then clear from (16.3d), (16.3c), and (16.7c). □

(16.8) REMARKS. (a) For a finitely realizable f there is up to scalar multiplication a *unique* algebraic difference equation E with the two properties: (i) E is of minimal order r, and (ii) E is irreducible as a polynomial in $k[L_0, L_1, \ldots, L_r, S_r, \ldots, S_1]$. Uniqueness follows easily from the discussion in HODGE and PEDOE [1968, page 110].

(b) It is easily verified that two polynomial input/output maps satisfying the same rational equation necessarily coincide. Moreover, by simple division of polynomials, f (more precisely, ψ_f) can be reconstructed from the rational equation E.

(c) It can be seen by a counterexample that Σ_f = polynomial system does not imply that f satisfies an integral difference equation. □

17. Jacobian Condition.

We now show how to check the condition "f is finitely realizable" by examining increasing truncations of the Volterra series. We take k to be a field of zero characteristic (when $\operatorname{char} k = p \neq 0$ one may generalize the criterion via spaces of differentials).

As in the previous section, we assume $p = 1$ without loss of generality.

Denote by D_{rs}, $1 \leq r \leq m$, $s \geq 1$, the operator which takes partial derivatives of polynomials with respect to the indeterminate ξ_{rs}. Let $D_s P$, $s \geq 1$, be the row vector $(D_{1s}P, \ldots, D_{ms}P)$.

(17.1) DEFINITION. *The n-th Jacobian matrix* $J_n(f)$ *of* f *is*

$$J_n(f) := \begin{pmatrix} D_1\epsilon_n\psi_f & D_2\epsilon_n\psi_f & \cdots & D_n\epsilon_n\psi_f \\ D_1\epsilon_n\psi_f(S_1) & \cdot & \cdots & D_n\epsilon_n\psi_f(S_1) \\ \vdots & \vdots & \cdots & \vdots \\ D_1\epsilon_n\psi_f(S_{n-1}, \ldots, S_1) & \cdot & \cdots & D_n\epsilon_n\psi_f(S_{n-1}, \ldots, S_1) \end{pmatrix}.$$

(17.2) EXAMPLE. Let f be a linear response map with "impulse response" $A_1, \dots, A_n, \dots$, i.e. $\psi_f = \sum a_{ij}\xi_{ij}$ and $A_i = (a_{1j}, \dots, a_{mj})$, $j \geq 1$. Then

$$\epsilon_n\psi_f(S_r, \dots, S_1) = \sum_{j=1}^{r}\sum_{i=1}^{m} a_{ij}S_{i,r-j} + \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i,j+r}\xi_{ij},$$

so

$$D_s\epsilon_n\psi_f(S_r, \dots, S_1) = A_{s+r}.$$

Thus $J_n(f)$ is the n-th principal minor of the block Hankel matrix of f (KALMAN, FALB, and ARBIB [1969, Chapter 10]), and it is well-known that realizability of f is equivalent to the existence of an integer s such that rank $J_n(f) < s$ for all n. The Jacobian of f gives a new way of interpreting the classical Hankel matrix of f. □

(17.3) THEOREM. f *is finitely realizable if and only if there exists an* $s \geq 0$ *such that*

$$\text{rank } J_t(f) \leq s \quad \textit{for all} \quad t \geq 0,$$

i.e., if and only if every (s + 1)-*minor of* $J_t(f)$ *is zero for all* $t \geq 0$.

PROOF. By HODGE and PEDOE [1968, III.7, Theorem III],

$$\text{rank } J_n(f) = \text{trdeg}_K K[\epsilon_n\psi_f, \dots, \epsilon_n\psi_f(S_{n-1}, \dots, S_1)].$$

(K is $k(\{S_{ij}\})$ as in the previous section.)

["only if"] By (10.12h), there is an s such that $\text{trdeg}_k \underset{\sim}{A}_f^n \leq s$ for all $n \geq 0$. By the Main Lemma (10.7) each $\epsilon_n\psi_f(S_{n-1}, \dots, S_1)$, $j = 0, \dots, n-1$ is a K-linear combination of elements of $\underset{\sim}{A}_f^n$. So rank $J_n(f) \leq s$.

["if"] If rank $J_t(f) \leq s$ for all $t \geq 0$, in particular

$$(17.4) \quad \operatorname{trdeg}_K K[\epsilon_t\psi_f, \ldots, \epsilon_t\psi_f(S_s, \ldots, S_1)] \leq s,$$

for all $t \geq s$. Let B be the subalgebra of Ψ_K generated by $\psi_f, \ldots, \psi_f(S_s, \ldots, S_1)$. By (17.4), $\operatorname{trdeg}_K \epsilon_t(B) \leq s$ for all $t \geq s$. It follows from (5.22) that $\operatorname{trdeg}_K B \leq s$. So there is an $r \leq s$ such that $\psi_f(S_r, \ldots, S_1)$ is algebraically dependent over $\psi_f, \ldots, \psi_f(S_{r-1}, \ldots, S_1)$, and the conclusion follows from (16.7a). $\square$

(17.5) **EXAMPLE.** As an application of (17.4), we prove that the polynomial response map f, with $m = p = 1$ and

$$\psi_f := \xi_{11} + \xi_{12}^2 + \xi_{13}^3 + \xi_{14}^4 + \xi_{15}^5 + \ldots ,$$

is not finitely realizable. Since

$$\epsilon_n\psi_f(S_r, \ldots, S_1) = S_r + S_{r-1}^2 + \ldots + S_1^r + \xi_{11}^{r+1} + \ldots + \xi_{1n}^{r+n},$$

for any $s \leq n$, $D_s\epsilon_n\psi_f(S_r, \ldots, S_1) = (r + s)\xi_{1s}^{r+s-1}$; this latter expression is also $D_s\epsilon_{n-1}\psi_f(S_r, \ldots, S_1)$ if $s < n$. Therefore

$$J_n(f) = \left(\begin{array}{c:c} J_{n-1}(f) & * \\ \hdashline * & (2n - 1)\xi_{1n}^{2n-2} \end{array}\right),$$

and so

$$\det J_n(f) = (2n - 1) \det J_{n-1}(f)\xi_{1n}^{2n-2} + t(\xi_{11}, \ldots, \xi_{1n}),$$

where $\deg_{1n} t < 2n - 2$. Since ξ_{1n} is algebraically independent over $k(\xi_{11}, \ldots, \xi_{1,n-1})$, an induction on n shows that $\det J_n(f) \neq 0$ for all n. $\square$

18. Some Examples and Counterexamples.

We discuss now several examples related to canonical realizations and input/output equations.

(18.1) **EXAMPLE.** We wish to illustrate the calculation of a nonpolynomial canonical system, via (10.4). Consider the system $\Sigma_o := (k^2, P, h, 0)$, where $m = p = 1$ and the equations of Σ_o are

$$\begin{aligned} x_1(t+1) &= x_1(t) + u(t), \\ x_2(t+1) &= x_1(t)x_2(t) + x_1(t) + x_2(t), \\ y(t) &= x_2(t). \end{aligned}$$

The 2-step reachability map of Σ_o is

$$g_2 : U^2 \to k^2 : (u_2, u_1) \mapsto \begin{pmatrix} u_1 + u_2 \\ u_2 \end{pmatrix}, \tag{18.2}$$

which is obviously onto. So Σ_o is reachable.

We want to calculate a canonical realization of $f_o := f_{\Sigma_o}$. Since Σ_o is reachable, $\Sigma_{f_o} \simeq \Sigma_o^{obs}$; we apply the construction in (10.4) to obtain Σ_{f_o}. The observation algebra $\underset{\sim}{A}(\Sigma_o)$ will be determined as $\bigcup_{t\geq 0} A^{0,t}(\Sigma_o)$ by induction on t, using the Main Lemma (10.7). Write $A(X) = k[\eta_1, \eta_2]$, so

$$A(P) : \eta_1 \mapsto \eta_1 + T, \quad \eta_2 \mapsto \eta_1\eta_2 + \eta_1 + \eta_2,$$

and

$$h = \eta_2.$$

To simplify calculations, let $\hat{\eta}_i := \eta_i + 1$, $i = 1, 2$, so that $A(P) : \hat{\eta}_1 \mapsto \hat{\eta}_1 + T$, $\hat{\eta}_2 \mapsto \hat{\eta}_1\hat{\eta}_2$ and $h = \hat{\eta}_2 - 1$. Then $\underset{\sim}{A}^{0,0} = k[\hat{\eta}_2 - 1] = k[\hat{\eta}_2]$. By induction,

$$\underset{\sim}{A}^{0,t} = k[\hat{\eta}_2, \hat{\eta}_2\hat{\eta}_1, \ldots, \hat{\eta}_2\hat{\eta}_1^t],$$

because

$$\begin{aligned} A(P)(\hat{\eta}_2\hat{\eta}_1^t) &= A(P)(\hat{\eta}_2).(A(P)(\hat{\eta}_1))^t \\ &= \hat{\eta}_1\hat{\eta}_2(\hat{\eta}_1 + T)^t \end{aligned}$$

$$= \sum_{j=0}^{t} \binom{t}{j} \hat{\eta}_2 \hat{\eta}_1^{j+1} T^{t-j},$$

so the new generator $\hat{\eta}_2 \hat{\eta}_1^{t+1}$ appears as the coefficient of T^o. Thus

$$(18.3) \quad \underset{\sim}{A}_{f_o} = \underset{\sim}{A}(\Sigma_o) = k[\{(\eta_2 + 1)(\eta_1 + 1)^t, \quad t \geq 0\}]$$

is not a finitely generated algebra. Therefore Σ_{f_o} is not a polynomial system.

By (3.17) the canonical state space $X_{f_o} = X(\underset{\sim}{A}_{f_o}) = X(k[\hat{\eta}_2 \hat{\eta}_1^t, t \geq 0])$ is an almost-variety which can be represented by the principal open set $D := \{\eta_2 \neq -1\}$ in $k^2 = X(k[\eta_1, \eta_2])$ (or, equivalently, $\{\hat{\eta}_2 \neq 0\}$ in $X(k[\hat{\eta}_1, \hat{\eta}_2])$) plus an extra point $\{*\}$; see (3.19). The morphism

$$T: X \to X^{obs} = X_{f_o},$$

in (10.4) is the map $X(i)$, where $i: A_{f_o} \to k[\hat{\eta}_1, \hat{\eta}_2]$ is the inclusion map. Therefore

$$T(x) = x \quad \text{if} \quad x \quad \text{is in} \quad D, \quad \text{and}$$

$$T(x) = * \quad \text{if} \quad x_2 = -1.$$

Since T is a k-system morphism, the transitions in Σ_{f_o} are

$$P_{f_o}(x, u) = T(P(x, u)) \quad \text{if} \quad x \quad \text{is in} \quad D, \quad \text{and}$$

$$P_{f_o}(*, u) = * \quad \text{for all} \quad u \quad \text{in} \quad U.$$

The initial state in Σ_{f_o} is 0 in D and the output is given by

$$h(x) = x_2 \quad \text{if} \quad x \quad \text{is in} \quad D, \quad \text{and}$$

$$h(*) = -1.$$

Note that T identifies precisely those states of X_{Σ_o} which are indistinguishable.

(18.4) EXAMPLE. Consider Σ_o and f_o as in (18.1). We now prove that *there exists no polynomial realization of* f_o *in which every pair of reachable states is distinguishable*. In other words, it is impossible to find any polynomial system $\hat{\Sigma}$ realizing f_o and a one-to-one abstract system morphism

$$T_{ac}: \Sigma_{ac} \to \hat{\Sigma},$$

where Σ_{ac} is the abstractly canonical realization of f_o. Our claim depends on two facts. (i) Let $A := A(g_2)(\underline{\underline{A}}(\Sigma_o)) \subseteq A(U^2)$. Then A *is maximally separating* (1.11) *with respect to* $A(U^2)$. Indeed, write $A(U^2) = k[T_1, T_2]$; by (18.2) and (18.3),

$$A = k[\{(T_2 + 1)(T_1 + T_2 + 1)^t, \quad t \geq 0\}].$$

With a change of variables $\hat{T}_2 := T_2 + 1$, $\hat{T}_2 := T_1 + T_2 + 1$ in $A(U^2)$, A becomes $k[\{T_2T_1^t, \ t \geq 0\})$. By (1.12), A is maximally separating. (ii) Now our claim follows from the following more general fact:

(18.5) LEMMA. *Let* f *be any polynomial response map. Assume that* (i) *the* n-*step reachability map* $g_{f,n}$ *of* Σ_f *is onto and* (ii) $A(g_{f,n})(X_f)$ *is maximally separating in* $A(U^n)$. *Then any quasi-reachable* Σ *realizing* f *whose reachable states are distinguishable is isomorphic to* Σ_f.

PROOF. Let $T: \Sigma \to \Sigma_f$ be the unique dominating morphism (10.4). We must prove that T is an isomorphism. The restriction of T to the reachable part Σ_R of Σ induces a morphism of *abstract* systems $T_R: \Sigma_R \to \Sigma_f$. Since by hypothesis Σ_R is abstractly canonical, T_R is a bijection (7.8). Therefore Σ_R is also reachable in n steps. In particular, Σ is quasi-reachable in n steps and $A(g_n)$ is one-to-one. Let $B := A(g_n)(X_\Sigma)$. Since $T \circ g_n = g_{f,n}$, T is an isomorphism iff $B = A := A(g_{f,n})(X_f)$. By definition of "maximally separating" (1.11) it will be enough to prove that B separates no more points of U^n than A. So take v, w in U^n and suppose that A does not separate v and w, i.e.

(18.6) $a(v) = a(w)$ for all a in A.

By definition of $A(g_{f,2})$, (18.5) is equivalent to

$$c(g_{f,n}(v)) = c(g_{f,n}(w)) \quad \text{for all } c \text{ in } \underset{\sim}{A}_f,$$

in other words, $g_{f,n}(v) = g_{f,n}(w)$. Since $g_{f,n} = T \circ g_n$ and T is one-to-one on reachable states, we conclude that $g_n(v) = g_n(w)$. So $d(g_n(v)) = d(g_n(w))$ for all d in $A(X)$, which means that $b(v) = b(w)$ for all b in B, i.e. neither does B separate v and w. □

When $k = \underline{\underline{R}}$ or k is algebraically closed the hypothesis of Lemma (18.6) can be weakened considerably, replacing (i) by just $\dim \Sigma_f = n$. The (easy) proof of this stronger lemma uses (4.6) plus (18.6).

(18.7) **EXAMPLE.** We continue to investigate Σ_o, f_o, and determine the (unique) irreducible equation of minimal order satisfied by f_o.

Let $\hat{\eta}_1$, $\hat{\eta}_2$ be coordinates for Σ_o defined as before. Working in $\underset{\sim}{Q}(\Sigma_o)$,

$$h + 1 = \hat{\eta}_2, \quad h^{u_2} + 1 = \hat{\eta}_1\hat{\eta}_2 \quad \text{and} \quad h^{u_2u_1} + 1 = \hat{\eta}_1\hat{\eta}_2 + \hat{\eta}_1\hat{\eta}_2u_2,$$

for all (u_2, u_1) in U^2. Since

$$\hat{\eta}_1^2\hat{\eta}_2 = (\eta_1\eta_2)^2/\eta_2 = (h^{u_2} + 1)^2/(h + 1),$$

we conclude that

$$(h + 1)(h^{u_2u_1} + 1) - (h^{u_2} + 1)^2 - (h^{u_2} + 1)(h + 1)u_2 = 0,$$

for all (u_2, u_1) in U^2. Since $h(g(w)) = f(w)$ for all w in $U[z]$, the polynomial

$$E := (L_2 + 1)(L_o + 1) - (L_1 + 1)^2 - (L_1 + 1)(L_2 + 1)S_2,$$

gives an algebraic (rational) difference equation for f_o, i.e.

$$[y(t - 2) + 1][y(t) + 1] - [y(t - 1) + 1]^2 - \\ - [y(t - 1) + 1][y(t - 2) + 1]u(t - 2) = 0,$$

for all input/output pairs (y, u) of $\underline{\underline{f}}_o$. The polynomial E is easily seen to be irreducible, and E is of minimal order for f_o, because $h = \hat{\eta}_2 - 1$ and $h^u = \hat{\eta}_1\hat{\eta}_2 - 1$ are algebraically independent. $\square$

(18.8) EXAMPLE. Since always $\operatorname{trdeg}_K \mathbf{Q}_f^K \leq \operatorname{trdeg} \mathbf{Q}_f$, there is always an algebraic difference equation of degree $\dim \Sigma_f$. *There may exist*, however, *equations of order strictly lower than* $\dim \Sigma$. To illustrate this, consider the bilinear input/output map $f_1 = f_{\Sigma_1}$, where $m = 2$, $p = 1$, and $\Sigma_1 := (k^{2n+1}, P, h, 0)$, $n \geq 1$ arbitrary, and P, h are given by the equations

$$\begin{aligned} x_1(t + 1) &= u_1(t), & x_{n+1}(t + 1) &= u_2(t), \\ x_2(t + 1) &= x_1(t), & x_{n+2}(t + 1) &= x_{n+1}(t), \\ &\vdots & &\vdots \\ x_n(t + 1) &= x_{n-1}(t), & x_{2n}(t + 1) &= x_{2n-1}(t), \end{aligned}$$
$$x_{2n+1}(t) = x_n(t)u_2(t) + x_{2n}(t)u_1(t),$$
$$y(t) = x_{2n+1}(t).$$

It is easily shown that Σ_1 is canonical, so $\dim \mathbf{Q}_{f_1} = \dim \Sigma_1 = 2n + 1$. However, f_1 satisfies the (affine) difference equation

$$y(t + 1) = u_1(t - n)u_2(t) + u_2(t - n)u_1(t),$$

of order $n + 1 < 2n + 1$. Note that f_1 is also a counterexample to $\dim \mathbf{L}_f = \dim \mathbf{L}_f^K$. Counterexamples with $m = 1$ also exist; e.g. $X_{\Sigma_2} := k^3$, $x^\#_{\Sigma_2} := 0$ and Σ_2 given by

$$\begin{aligned}
x_1(t + 1) &= u(t),\\
x_2(t + 1) &= x_3(t),\\
x_3(t + 1) &= x_3(t)x_1(t) + x_1(t) + x_2(t)u(t),\\
y(t) &= x_3(t),
\end{aligned}$$

which satisfies an affine equation of order 2:

$$y(t) = y(t - 1)u(t - 2) + y(t - 2)u(t - 1) + u(t - 2).$$

(18.9) **EXAMPLE.** When Σ_f is a polynomial system, it is natural to represent it by a system of simultaneous polynomial difference equations. An important question in applications concerns the minimal possible number r of equations in such representations of a given Σ_f. By (3.4ff), r = smallest possible cardinality of a set of generators for Σ_f. In general, $r \leq \operatorname{trdeg} A_f$, and equality holds if and only if $A_f = k[T_1, \ldots, T_r]$, which is always the case for linear response maps f. A result of KALMAN [1976] shows that A_f is a polynomial ring also for bilinear single-output $(p = 1)$ response maps. For more general response maps, even bounded ones, X_f may be different from affine space. For instance, let $m = p = 1$ and let f_3 be the response map satisfying

$$y(t) = u^2(t - 2)u(t - 1) + u^3(t - 2).$$

Then $f_3 = f_{\Sigma_3}$, where $X_{\Sigma_3} := k^2$, $x^{\#}_{\Sigma_3} := 0$ and Σ_3 has equations

$$\begin{aligned}
x_1(t + 1) &= u(t),\\
x_2(t + 1) &= x_1^2(t)u(t) + x_1^3(t),\\
y(t) &= x_2(t).
\end{aligned}$$

Since Σ_3 is quasi-reachable, $\underset{\sim}{A}_f = \underset{\sim}{A}(\Sigma) = k[\eta_1^2, \eta_1^3, \eta_2]$. Thus to represent the canonical realization (polynomial by (16.2c)) Σ_{f_3} one needs at least 3 equations. (Note that the noncanonical realization Σ_3 of f_3 requires only 2 equations!) A representation of Σ_{f_3} is, for instance,

$$\begin{aligned} x_1(t+1) &= u^2(t), \\ x_2(t+1) &= u^3(t), \\ x_3(t+1) &= x_1(t)u(t) + x_2(t), \qquad x^\# = 0, \\ y(t) &= x_3(t), \end{aligned}$$

where $X_{f_3} = \{(x_1, x_2, x_3)$ in $k^3 \mid x_1^3 = x_2^2\}$, a surface with a singularity at $x = 0$. Since $\dim X_f = 2$, Σ_3 is minimal, a fact which agrees with Σ_3 being quasi-reachable and abstractly observable, so weakly canonical (13.7). □

(18.10) **EXAMPLE.** Consider a *homogeneous* polynomial response map f; i.e. all monomials in ψ_f have the same degree, say s. Take, for simplicity, $m = p = 1$. One way of applying the theory of *multilinear* response maps (15.3) for obtaining realizations of such f is the following. Let

$$\psi_f = \sum a_{i_1 \dots i_s} \xi_{1i_1} \cdots \xi_{1i_s},$$

where the sum runs over all possible sequences (with repetitions) $i_1, \dots, i_s$ of integers ≥ 1. Define the s-linear response map f^o (with $m = s$, $p = 1$) by

$$\psi_{f^o} := \sum a_{i_1 \dots i_s} \xi_{1i_1} \cdots \xi_{si_s}.$$

It is easily verified that f is finitely realizable iff f^o is finitely realizable. Given any realization Σ of f^o, a realization $\hat{\Sigma}$ of f is obtained by applying the same input to all the input channels of Σ. Such a method of realizing a homogeneous f has been suggested by several authors (see, for instance, BUSH [1965]).

The method is theoretically unsatisfactory, however, given the noncanonical nature of $\hat{\Sigma}$. The following example shows that this procedure may be also unsatisfactory from a practical (synthesis) point of view. Specifically, we shall give a homogeneous response map f of (arbitrary) degree s such that Σ_f has $X_f = k$ but such that the lowest possible dimension for a realization $\hat{\Sigma}$ obtained by the above procedure is s. That is, we look for an f with $X_f = k$ and $\dim \Sigma_{fo} = s$.

Let Σ_{f_4} have the equations (with $X = k$)

$$x(t+1) = x(t) + u(t), \quad x^{\#} := 0,$$

$$y(t) = x^s(t).$$

Then $f_4^o = f_{\Sigma_4}$, where Σ_4 has $m = s$, $p = 1$, $X = k^s$ and equations

$$x_i(t+1) = x_i(t) + u_i(t), \quad x_i^{\#} := 0, \quad i = 1, \ldots, s.$$

$$y(t) = x_1(t) \ldots x_s(t).$$

Clearly, Σ_4 is reachable. Moreover, Σ_4 is algebraically observable, since, writing $A(k^s) = k[\eta_1, \ldots, \eta_s]$, the coefficient of $T_1 \ldots \hat{T}_i \ldots T_s$ (T_i omitted) in

$$A(P)h = (\eta_1 + T_1) \ldots (\eta_s + T_s),$$

is η_i. Thus Σ_4 is canonical and has dimension s. □

(18.11) EXAMPLE. Consider strictly recursive equations

$$(*) \qquad y(t) = R(y(t-1), \ldots, y(t-r), u(t-1), \ldots, u(t-r)),$$

for finitely realizable polynomial responses f, where R is a polynomial. Such equations are well-known to exist for linear response maps. By the Cayley-Hamilton Theorem, the internally-bilinear response maps (15.4) of systems of the form

$$x(t+1) = Fx(t)u(t),$$

$$y(t) = Hx(t),$$

($m = 1$, F, H = linear) are easily seen to also satisfy equations (*).

In general, however, a finitely realizable polynomial response map satisfies no equation such as (*). To construct counterexamples it is enough to exhibit systems Σ such that for each $r \geq 1$ there exist pairs of inputs $w, \hat{w}$ for which $w(t) = \hat{w}(t)$ for $t \geq 0$, $\underline{\underline{f}}(w)(t) = \underline{\underline{f}}(\hat{w})(t)$ for $0 \leq t < r$ but $\underline{\underline{f}}(w)(r) \neq f(\hat{w})(r)$. Such $w, \hat{w}$ clearly contradict (*). We give three such counterexamples.

(a) A one-dimensional system Σ_5 with $m = p = 1$, where

$$x(t + 1) = x(t) + u(t), \quad x^{\#} := 0,$$
$$y(t) = x^2(t).$$

Here take $w(t) = \hat{w}(t) := 0$ if $t \neq -1,\ r - 1$, $w(r - 1) = \hat{w}(r - 1) := 1$, $w(-1) := -1$ and $\hat{w}(-1) := 1$.

(b) Again $m = p = 1$, and Σ_6 given by

$$x_1(t + 1) = x_1(t) + x_1(t)u(t), \quad x_1^{\#} := 1,$$
$$x_2(t + 1) = x_2(t) + x_2(t)u(t), \quad x_2^{\#} := 1,$$
$$y(t) = x_1(t) + x_2(t).$$

Here take $w(t) = \hat{w}(t) := 0$ if $t \neq -1,\ r - 1$, $w(r - 1) = \hat{w}(r - 1) := \binom{1}{-1}$, $w(-1) := 0$ and $\hat{w}(-1) := \binom{1}{-1}$. Note that f_{Σ_6} is an internally-bilinear response map.

(c) Let $m = 2$, $p = 1$, and Σ_7 given by

$$x_1(t + 1) = x_1(t) + u_1(t), \quad x_1^{\#} := 0,$$
$$x_2(t + 1) = x_2(t) + u_2(t), \quad x_2^{\#} := 0,$$
$$y(t) = x_1(t)x_2(t).$$

Here take $w(t) = \hat{w}(t) := 0$ if $t \neq -1,\ r - 1$, $w(r - 1) = \hat{w}(r - 1) = \hat{w}(-1) := \binom{1}{1}$ and $w(-1) := -\binom{1}{1}$. Note that f_{Σ_7} is a bilinear response map. □

CHAPTER V. STATE-AFFINE SYSTEMS

Some special types of system configurations arose in Theorem (15.5) as natural realizations for bounded input/output maps. The most useful of these configurations are state-affine systems; they are studied in this chapter.

As indicated in Chapter I, we shall base our development on the notion of a representation of the Volterra (or, equivalently, the exponent) series of f.

19. *Recognizable Series*.

Throughout this section, φ is an exponent series (5.6) whose support is contained in Δ_J, where

$$J = \{\delta_o = (0),\ \delta_1,\ \ldots,\ \delta_s\}.$$

(19.1) DEFINITION. *A representation (with support in* Δ_J*) is an object* $R = (X,\ \{F_\alpha,\ \alpha \text{ in } J\},\ \{g_\alpha,\ \alpha = \delta_1,\ \ldots,\ \delta_s\},\ h)$, *where*

(a) X *is a vector space over* k,

(b) *each* $F_\alpha\colon X \to X$ *is a linear map*,

(c) *each* g_α *is in* X, *and*

(d) $h\colon X \to Y = k^p$ *is a linear map*.

For each $\alpha = \alpha_1\ldots\alpha_t$ *in* J^* *let* $F_\alpha := F_{\alpha_1}\ldots F_{\alpha_t}$, *and write* $F_\Lambda := 1_X$. *If* $\gamma = \alpha_1\ldots\alpha_t(0)\ldots(0)$ *is in* J^*, *with* $\alpha_t \neq (0)$, *then*

$$g_\gamma := F_{\alpha_1\ldots\alpha_{t-1}} g_{\alpha_t};$$

if $\gamma = (0)\ldots(0)$, $g_\gamma := 0$. *With these notations*, R *is accessible iff*

$$\text{span } \{g_\alpha,\ \alpha \text{ in } \Delta_J\} = X;$$

R *is reduced iff*

$$\bigcap_{\alpha \text{ in } J^*} \ker hF_\alpha = \{0\};$$

R *is canonical iff* R *is both accessible and reduced. The dimension of* R *is*

$$\dim R := \dim X \leq \infty.$$

The exponent series φ_R *represented by* R *is given by*

$$\varphi_R(\alpha) := hg_\alpha \quad \underline{\text{for all}} \quad \alpha \quad \underline{\text{in}} \quad \Delta_J.$$

The representation R *is minimal iff, for any* $\hat{R}$ *for which* $\varphi_R = \varphi_{\hat{R}}$, *necessarily* $\dim R \leq \dim \hat{R}$. *When* $\dim R < \infty$, φ_R *is recognizable*. *The linear map* $T\colon X \to \hat{X}$ *induces a morphism of representations*

$$T\colon R = (X, \{F_\alpha\}, \{g_\alpha\}, h) \to \hat{R} = (\hat{X}, \{\hat{F}_\alpha\}, \{\hat{g}_\alpha\}, \hat{h}),$$

iff $Tg_\alpha = \hat{g}_\alpha$ *and* $T \circ F_\alpha = \hat{F}_\alpha \circ T$ *for all* α *and* $h = \hat{h} \circ T$.

The terminology "recognizable" is taken from automata theory, as explained in Chapter I.

It is easy to see that representations form a category with the above notion of morphism.

(19.2) DEFINITION. *The behavior matrix* $\underline{\underline{B}}(\varphi)$ *of* φ *is an infinite block matrix, with rows indexed by* J^* *and columns indexed by* Δ_J, *whose* (α, β)*-th entry is* $\varphi(\alpha\beta)$, *a column vector in* $Y = k^p$.

We denote by B_β the β-th column of $\underline{\underline{B}}(\varphi)$.

(19.3) THEOREM. *Any* φ *has a canonical representation* R_φ. *If* R *and* $\hat{R}$ *are two canonical representations of* φ, *there exists a unique representation isomorphism* $T\colon R \to \hat{R}$. *Further*, φ *is recognizable if and only if* $\operatorname{rank} \underline{\underline{B}}(\varphi) < \infty$; *in this case a representation* R *of* φ *is canonical* (i) *if and only if* R *is minimal and* (ii) *if and only if* $\dim R = \operatorname{rank} \underline{\underline{B}}(\varphi)$.

PROOF. We define $R_\varphi = (X, \{F_\alpha\}, \{g_\alpha\}, h)$ as follows:

X := linear space spanned by $\{B_\beta, \ \beta \text{ in } \Delta_J\}$,

$g_{\delta_i} := B_{\delta_i}$, $i = 1, \ldots, s$,

h := linear map induced by the projections in the Λ-th block row: $B_\beta \mapsto \varphi(\beta)$, and

F_{δ_i} := linear map induced by the column shifts $B_\beta \mapsto B_{\delta_i \beta}$, $i = 0, \ldots, s$.

Note that the F_α are well-defined, since any relation among columns

(19.4) $\quad \sum_{(\text{finite})} r_\beta B_\beta = 0, \quad r_\beta \text{ in } k,$

implies $\sum r_\beta B_{\delta_i \beta} = 0$, $i = 0, \ldots, s$. Indeed, the α-th (block) row of $\sum r_\beta B_{\delta_i \beta}$ is $\sum r_\beta \varphi(\alpha\delta_i\beta)$, which is also the $\alpha\delta_i$-th row of (19.4), and hence zero.

Clearly,

$$g_\beta = F_{\beta_1 \ldots \beta_{t-1}} g_{\beta_t} = B_\beta,$$

for any β in Δ_J, so R is accessible. By definition of h,

$$\varphi_{R_\varphi}(\beta) = h(g_\beta) = h(B_\beta) = \varphi(\beta),$$

so R_φ represents φ. Now take $x = \sum r_\beta B_\beta$ in $\bigcap_\alpha \ker hF_\alpha$. Then

$$0 = hF_\alpha x = \sum r_\beta hF_{\alpha\beta_1 \ldots \beta_{t-1}} g_{\beta_t} = \sum r_\beta \varphi(\alpha\beta) = \alpha\text{-th block row of } x,$$

for all α in J^*. Thus $x = 0$ and R is reduced, hence canonical.

The proof of the theorem will be complete after we establish the following lemma (similar to (7.7) and (11.3)):

(19.5) LEMMA. *Let* $R, \hat{R}$ *be representations of* φ, *with* R *accessible and* $\hat{R}$ *reduced. Then there exists a unique representation morphism*

$T: R \to \hat{R}$. *When* $\hat{R}$ *is canonical,* T *is onto*.

PROOF. Define $T: X \to \hat{X}$ as the linear extension of

$T(g_\alpha) := \hat{g}_\alpha$, α in Δ_J.

By definition of representation morphism, this is clearly the only possible choice for T. Thus the lemma will follow if T is well-defined. We must show that if $x = \sum r_\beta g_\beta = 0$ (finite sum) then also $\hat{x} := \sum r_\beta \hat{g}_\beta = 0$. Pick α in J^*. Then

$$\hat{h}\circ\hat{F}_\alpha(\hat{x}) = \sum r_\beta \hat{h}(\hat{g}_{\alpha\beta}) = \sum r_\beta \varphi(\alpha\beta) = \sum r_\beta h(g_{\alpha\beta}) = hF_\alpha(x) = 0.$$

Since $\hat{R}$ is reduced, this means that $\hat{x} = 0$. □

(19.6) REMARK. The above theorem gives rise to an algorithm for constructing representations of a recognizable series φ from $\underline{\underline{B}}(\varphi)$. It is only necessary for this purpose *to find a submatrix* Φ *of full rank of* $\underline{\underline{B}}(\varphi)$ and to express the g_α, F_α and h with respect to the basis consisting of the set of columns used in the definition of Φ. This algorithm is a minor variation of that given by FLIESS [1972] and generalizes the one given for the linear case by ROUCHALEAU [1972]. □

(19.7) EXAMPLE. Let $m = p = 1$ and take a linear (15.2) response map f, with $\psi_f = \sum a_i \xi_{1i}$, $\varphi_f = (\sum a_i 0^i)1$. The only possible nonzero columns of $\underline{\underline{B}}(\varphi_f)$ are those indexed by $1, 01, \ldots,$ and the only possible nonzero rows are those indexed by $\{0^n,\ n \geq 0\}$:

$$\begin{array}{c|ccccccc}
 & 1 & 01 & 0^2 1 & \cdots & 0^n 1 & \cdots \\
\hline
\Lambda & a_1 & a_2 & a_3 & \cdots & a_{n+1} & \cdots \\
0 & a_2 & a_3 & a_4 & \cdots & a_{n+2} & \cdots \\
0^2 & a_3 & a_4 & a_5 & \cdots & a_{n+3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & & & \\
0^n & a_{n+1} & \cdot & \cdot & & & \\
\vdots & \vdots & \vdots & \vdots & & &
\end{array},$$

the classical *Hankel matrix* of the linear response map f. When $m, p \neq 1$, the only possible nonzero part of $\underline{\underline{B}}(\varphi)$ becomes, with a suitable ordering of row and column indexes, the block Hankel matrix of f; the realization procedure in (16.6) coincides with the well-known "Silverman's formulas" (SILVERMAN [1971]) □

(19.8) EXAMPLE. When f is internally-bilinear (15.4), our $\underline{\underline{B}}(\varphi)$ becomes the generalized Hankel matrix introduced by ISIDORI [1973] and FLIESS [1973]. When f is bilinear in the input/output sense (15.3), the nonzero part of $\underline{\underline{B}}(\varphi)$ can be arranged so as to become the matrix introduced by KALMAN [1976]. □

20. *State-Affine Systems*.

(20.1) DEFINITION. *The polynomial system* Σ *is state affine iff*

(a) $X = k^n$ *for some integer* n,

(b) $h: X \to k^p$ *is a linear map*,

(c) $x^\# = 0$, *and*

(d) *for each fixed input value* u, $P(x, u)$ *is an affine* (= *linear* + *translation) function of* x.

In other words, there exist polynomials $p_{ij}(T_1, \ldots, T_m)$ *and* $q_i(T_1, \ldots, T_m)$ *such that the transition equations for* Σ *are given by*

$$x_i(t + 1) = \sum_{j=0}^{n} p_{ij}(u(t))x_j(t) + q_i(u(t)), \quad i = 1, \ldots, n.$$

Σ *is span-reachable iff the set of reachable states* X_R *spans* k^n; *span-canonical iff both span-reachable and observable; minimal iff* $\dim \Sigma \leq \dim \hat{\Sigma}$ *for any state-affine system* $\hat{\Sigma}$ *for which* $f_\Sigma = f_{\hat{\Sigma}}$. *A system morphism* $T: \Sigma \to \hat{\Sigma}$ *between state-affine systems is a morphism of state-affine systems iff* $T: X \to \hat{X}$ *is linear*.

(20.2) REMARK. It follows from Theorem (15.5) that a polynomial response map f is bounded and finitely realizable iff f can be realized by a system which is state affine except for the condition $x^\# = 0$. A coordinate translation may of course be used to make $x^\# = 0$, without changing the affine form of P. However, this makes h an affine, rather than a linear, map. In other words, the equilibrium-level output $h(x^\#)$ may become nonzero. *We shall assume* for the rest of this chapter that a coordinate translation has been performed (if necessary) on $Y = k^p$ so that

$$\varphi_f(\Lambda) = 0 \quad \underline{\text{for all response maps}} \quad f.$$

With this convention, $h(x^\#) = 0$ for every system. Thus bounded + finitely realizable = realizable by a state-affine system. □

(20.3) REMARK. The basic observables of a state-affine system are affine functions of the state; thus

(abstract) observability = algebraic observability,

for such systems. □

Let Σ be a state affine system. Since P is affine in x and polynomial in u, there exists a subset $J = \{\delta_0 = (0),\ \delta_1,\ \ldots,\ \delta_s\}$ of $\underline{\underline{N}}^m$ and matrices $\{F_{\delta_i},\ \delta_i \text{ in } J\}$ and $\{g_{\delta_i},\ i = 1, \ldots, s\}$ such that

$$(20.4) \qquad P(x, u) = \sum_{i=0}^{s} F_{\delta_i} x u^{\delta_i} + \sum_{i=1}^{s} g_{\delta_i} u^{\delta_i}.$$

We shall call $R = R_\Sigma := (k^n,\ \{F_{\delta_i},\ i = 0, \ldots, s\},\ \{g_{\delta_i},\ i = 1, \ldots, s\},\ h)$ the *representation associated to* Σ. Take $w = w_t \ldots w_1$ in U^t. An easy calculation shows that

$$(20.5) \qquad P^{(t)}(x, w) = \sum_{\alpha \text{ in } J^t} (F_\alpha x + g_\alpha) w^\alpha.$$

Thus the t-step reachability map $g_t \colon U^t \to k^n$ becomes

$$(20.6) \quad g_t(w) = P^{(t)}(0,\ w) = \sum_{\alpha \text{ in } J^t} g_\alpha w^\alpha ,$$

and therefore

$$f_\Sigma(w) = h(g_t(w)) = \sum h(g_\alpha)w^\alpha = \sum \varphi_R(\alpha)w^\alpha ,$$

so that $\varphi_f = \varphi_R$.

We now have proved the following

(20.7) LEMMA. The assignment $\Sigma \to R_\Sigma$ is a bijection between state affine realizations of a bounded polynomial response map f and finite-dimensional respresentations (with fixed basis for X) of φ_f. □

The above assignment preserves reachability and observability properties:

(20.8) LEMMA. Σ is span-reachable [respectively observable] if and only if R_Σ is accessible [respectively reduced].

PROOF. Let $b\colon k^n \to k$ be a linear function, and take w in U^t. Then (19.6) implies that

$$(20.9) \quad b(g_t(w)) = \sum_{\alpha \text{ in } J^t} b(g_\alpha)w^\alpha .$$

Since k is an infinite field, the right side of (20.9) is zero for all w in U^t iff $b(g_\alpha) = 0$ for all α in J^t. Thus there exists a $b \neq 0$ with $b(g_t(w)) = 0$ for all w in $U[z]$ (i.e., Σ is not span-reachable) iff there is a $b \neq 0$ with $b(g_\alpha) = 0$ for all α (i.e., R_Σ is not accessible).

We now prove the observability part. Take w in U^t. Then

$$(20.10) \quad h^w(x) = h \circ P^{(t)}(x,\ w) = \sum_{\alpha \text{ in } J^t} h(F_\alpha x + g_\alpha)w^\alpha .$$

Two states x and $\hat{x}$ are indistinguishable iff $h^w(x) = h^w(\hat{x})$ for all w in U^*, which by (20.10) means that

$$(20.11) \quad \sum_{\alpha \text{ in } J^t} hF_\alpha(x - \hat{x})w^\alpha = 0 \ \ \underline{\text{for all}} \ \ w \ \ \underline{\text{in}} \ \ U^t \ \ \underline{\text{and all}} \ \ t \geq 0.$$

Using again the fact that k is infinite, (20.11) is equivalent to $x - \hat{x}$ being in $\ker hF_\alpha$, for all α in J^*. So h^Γ is one-to-one iff $\bigcap_\alpha \ker hF_\alpha = \{0\}$, as required. $\square$

(20.12) REMARK. The hypothesis k = infinite is essential for the above result in its present form. The correct generalization to arbitrary fields is that only monomials u^{δ_i} which are linearly independent as functions should be used in the definition of a state affine system. $\square$

From (20.7), (20.9), and (19.3) we obtain the main result of this section:

(20.13) THEOREM. *Any bounded and finitely realizable response map* f *has a span-canonical state-affine realization, unique up to isomorphism* (= *change of basis in the state-space). A realization* Σ *of* f *is span-canonical (i) if and only if* Σ *is minimal and (ii) if and only if* $\dim \Sigma = \operatorname{rank} \underline{\underline{B}}(\varphi_f)$. $\square$

Of course, realizations can be obtained from $\underline{\underline{B}}(\varphi_f)$ using the algorithm in (19.4). There is an interesting interpretation of the row-space $\underset{\sim}{R}_f$ of $\underline{\underline{B}}(\varphi_f)$, which implies a natural duality between the observation space $\underset{\sim}{L}_f$ and the state space of the span-canonical state-affine realization of f (i.e., the column space of $\underline{\underline{B}}(\varphi_f)$). Indeed, define

$$\eta: \underset{\sim}{L}_f \to \underset{\sim}{R}_f,$$

on generators by

$$(20.14) \quad \eta(f_j^w) := \sum_{\alpha \text{ in } J^t} \varphi_j(\alpha\beta)w^\alpha \ \ \text{for} \ \ w \ \ \text{in} \ \ U^t, \ \ j = 1, \ldots, p, \ \ t \geq 0.$$

It is not difficult to verify the following

(20.15) PROPOSITION. $\eta\colon \mathbf{L}_f \simeq \mathbf{R}_f$. *In particular*, $\dim \mathbf{L}_f = \operatorname{rank} B(\varphi_f) =$ $=$ *dimension of span-canonical state-affine realization of* f. $\square$

21. *Finite Response Maps and Cascades of Linear Systems*.

(21.1) DEFINITION. *The total degree of the polynomial response map* f *(or of its Volterra series* ψ_f*) is*

$$\operatorname{tdeg} f = \operatorname{tdeg} \psi_f := \sup \{\|\alpha\| \mid \psi_f(\alpha) \neq 0\} \leq \infty;$$

f *is finite iff* $\operatorname{tdeg} f < \infty$.

Since obviously $\deg \psi_f \leq \operatorname{tdeg} \psi_f$, a finite response map is in particular bounded.

(21.2) EXAMPLES. The nonzero polynomial response map f is *linear* if and only if $\operatorname{tdeg} f = 1$; when f is bilinear, $\operatorname{tdeg} f = 2$. On the other hand, an internally-bilinear response map f is bounded but not necessarily finite. $\square$

A natural class of realizations for finite maps will consist of feedback-free interconnections of linear systems, defined below:

(21.3) DEFINITION. *The polynomial system* Σ *is a cascade of linear systems iff* $X = k^n$, $x^\# = 0$ *and there exists a direct sum decomposition*.

$$X = X_1 \oplus \ldots \oplus X_d, \quad d \geq 1,$$

and linear maps $A_i\colon X_i \to X_i$, $i = 1, \ldots, d$ *such that the transition equations of* Σ *become*

$$x_i(t+1) = A_i x_i(t) + B_i(x_1(t), \ldots, x_{i-1}(t), u(t)),$$
$$x_i(t) \text{ in } X_i \ \textit{for all} \ t, \quad i = 1, \ldots, d.$$

Given a decomposition $X = X_1 \oplus \ldots \oplus X_d$ as above, let

(21.4) $Z_j := X_j \oplus \dots \oplus X_d, \quad j = 1, \dots, d.$

Conversely, for any chain $Z_d \subseteq Z_{d-1} \subseteq \dots \subseteq Z_1 = X$, there exists a decomposition $\{X_i\}$ such that (21.4) holds. Thus the following result is easy to prove.

(21.5) LEMMA. The state-affine system Σ is a cascade of linear systems if and only if there exists a chain of subspaces

$$\{0\} = Z_{d+1} \subseteq Z_d \subseteq \dots \subseteq Z_1 = X,$$

such that, with the notations of (20.4),

$$F_{\delta_o} Z_j \subseteq Z_j, \quad j = 1, \dots, d, \quad \text{and}$$

$$F_{\delta_i} Z_j \subseteq Z_{j+1}, \quad i = 1, \dots, s, \quad j = 1, \dots, d. \qquad \square$$

For any $\alpha = \alpha_1 \dots \alpha_t$ in $\operatorname{supp} \psi$, let

$$\sharp(\alpha) := \text{number of nonzero vectors among } \alpha_1, \dots, \alpha_t.$$

Clearly,

$$\sharp(\alpha) \leq \|\alpha\| \leq d.$$

The main result of this section is the following

(21.6) THEOREM. The following statements are equivalent for any polynomial response map f:

(a) f is bounded and the span canonical state-affine realization of f is a cascade of linear systems.

(b) f is realizable by a cascade of linear systems.

(c) f is a finitely realizable finite map.

PROOF. (a) $\Rightarrow$ (b) Trivial.

(b) $\Rightarrow$ (c) Let $f = f_\Sigma$, with Σ as in (21.3). Let each G_i have total degree s_i and let h have total degree s. It follows by induction on i that

$$\operatorname{tdeg} \psi_{f_\Sigma} \leq s s_1 \ldots s_n.$$

(c) $\Rightarrow$ (a) Let $\operatorname{tdeg} \psi_f = d$. Referring to the realization constructed via (19.3), let

$$Z_j := \operatorname{span} \{B_\beta, \ \beta \text{ in } \Delta_J, \ j \leq \sharp(\beta)\}, \quad j = 1, \ldots, d + 1.$$

The result is then clear by (21.5). □

22. Rationality.

We now indicate how the automata-theoretic notion of rationality generalizes to the present context. The convention (20.2) that $\varphi(\Lambda) = 0$ for every exponent series is still in force.

(22.1) DEFINITION. A scalar ($p = 1$) exponent series φ is rational iff φ can be expressed in terms of finitely many vectors in $\underline{\underline{N}}^m$ by means of a finite number of any of the following three types of operations:

(i) $(\varphi, \hat{\varphi}) \mapsto r\varphi + s\hat{\varphi}$, for any r, s in k,

(ii) $(\varphi, \hat{\varphi}) \mapsto \varphi\hat{\varphi}$,

(iii) $\varphi \mapsto \varphi^* := \sum_{i \geq 0} \varphi^i$,

where φ^* is interpreted as

$$\varphi^*(\alpha) := \sum_{i \leq |\alpha|} \varphi^i(\alpha), \quad \text{for all } \alpha \text{ in } \Delta_J.$$

A vector exponent series $(\varphi_1, \ldots, \varphi_p)'$ is rational iff each φ_j is rational.

(22.2) THEOREM (KLEENE-SCHÜTZENBERGER). The exponent series φ is rational if and only if φ is recognizable.

PROOF. See EILENBERG [1974, Theorem VII.5.1]. □

The notion of rationality can be interpreted in terms of a calculus of interconnections of state-affine systems. This calculus permits, via exponent series, the determination of the Volterra series of a given system and, conversely, the construction of a realization given (a rational expression for) a Volterra series. The calculus is obtained as a straightforward translation of the well-known manipulations with automata, as given for instance in EILENBERG [1974].

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